LECTURE 2. SCHUR'S LEMMA. EXAMPLES OF REPRESENTATIONS.

Please consider every proposition in this text to be an exercise.

1. Morphisms of representations and Schur's Lemma

Definition 1.1. Assume that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. $A \in \mathrm{Hom}(V, U)$ is called a morphism of representations T and S if

$$AT(g) = S(g)A, \quad \forall g \in G. \tag{1.1}$$

We are going to denote the set of morphisms of representations T and S by $\mathrm{Mor}(T, S)$ (by $\mathrm{End}(T)$ if $S = T$).

Question 1.2. How do we reformulate the definition of a morphism of representations if we consider those representations as G-modules?

Question 1.3. Is $\mathrm{End}(T)$ a subspace of $\mathrm{End}(V)$?

Exercise 1.4. Describe all morphisms between two one-dimensional complex representations of the group $\mathbb{Z}_n$.

Definition 1.5. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. They are called isomorphic if $\exists A \in \mathrm{Iso}(V, U)$ such that $A \in \mathrm{Mor}(T, S)$ (A is called an isomorphism of representations T and S in that case). The set of isomorphisms of representations T and S is denoted by $\mathrm{Iso}(T, S)$. The set of automorphisms of the representation T is denoted by $\mathrm{Aut}(T)$.

Lemma 1.6. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations, $A \in \mathrm{Mor}(T, S)$. Then*

a) $\ker A = \{v \in V \mid Av = 0\} \subset V$ *is an invariant subspace with respect to T;*

b) $\mathrm{im} A = \{Av \mid v \in V\} \subset U$ *is an invariant subspace with respect to S.*

Proposition 1.7. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are irreducible representations, $A \in \mathrm{Mor}(T, S)$. Then either $A = 0$ or $A \in \mathrm{Iso}(T, S)$.*

Proposition 1.8. *(Schur's Lemma) Suppose that $T : G \to \mathrm{GL}(V)$ is an irreducible complex representations, and $A \in \mathrm{End}(T)$. Then $\exists \lambda \in \mathbb{C}$ such that $A = \lambda \mathrm{Id}$.*

Question 1.9. Is the claim of Schur's Lemma still true over $\mathbb{R}$?

Definition 1.10. A vector space V over a field $\mathbb{F}$ with an operation $\cdot : V \times V \to V$ is called an algebra if the following properties hold:

(1) $a(b + c) = ab + ac, \quad \forall a, b, c \in V$;
(2) $(b + c)a = ba + bc, \quad \forall a, b, c \in V$;
(3) $\lambda(ab) = (\lambda a)b = a(\lambda b), \quad \forall a, b \in V, \forall \lambda \in \mathbb{F}$.

An algebra V is called associative if $a(bc) = (ab)c$, $\forall a, b, c \in V$. An associative algebra V is called a division algebra if $(V \setminus \{0\}, \cdot)$ is a group.

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Question 1.11. Let $T : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{F}$. Does $\mathrm{End}(T)$ have to be a division algebra over $\mathbb{F}$?

Question 1.12. What are possible finite-dimensional division algebras over $\mathbb{C}$?

Question 1.13. What are possible finite-dimensional division algebras over $\mathbb{R}$?

Exercise 1.14. Construct an irreducible real representation of a finite group such that its algebra of endomorphisms is isomorphic to the division algebra of quaternions. You are free to pick a finite group you prefer.

2. Symmetric Group

Definition 2.1. A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if ρ is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called a symmetric group and is denoted by S_n. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n). \end{pmatrix} \tag{2.1}$$

A more common notation is a cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{2.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{2.3}$$

One may check that it is indeed an equivalence relation. For each equivalence class that consists of, say, m elements we pick a representative j and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of ρ. If $m = 1$, we usually omit the corresponding one-element cycle. Here is an example of cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{2.4}$$

One may show that cycle notation is well-defined. A cycle of length 2 is called a transposition.

Here is an example of the product of two permutations written in cycle notation:

$$((1, 2)(3, 4, 5))((2, 3)(1, 5)) = (1, 3)(2, 4, 5). \tag{2.5}$$

Exercise 2.2. Show that S_n is a group with respect to composition and find its order.

One may compare the following statement with the change of basis for linear operators.

Exercise 2.3. Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$ the following equality holds

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{2.6}$$

where the LHS is the composition of three permutations such that the second permutation is a cycle of length k.

Exercise 2.4. Suppose that $\rho \in S_n$. Consider a map $\psi_\rho : S_n \to S_n$ defined by

$$\psi_\rho(\sigma) := \rho\sigma\rho^{-1}. \tag{2.7}$$

Show that ψ_ρ is an automorphism of the group S_n.

Exercise 2.5. Show that every permutation $\rho \in S_n$ can be represented as a product of some transpositions.

Proposition 2.6. *For every positive integer n there exists a unique map* $\operatorname{sgn} : S_n \to \{1, -1\}$ *such that*

(1) $\operatorname{sgn}((i,j)) = -1, \quad \forall 1 \le i < j \le n;$
(2) $\operatorname{sgn}(\rho\sigma) = \operatorname{sgn}(\rho)\operatorname{sgn}(\sigma).$

The value $\operatorname{sgn}(\rho)$ *is called the sign of the permutation* ρ.

To prove this proposition, one may consider the ratio

$$\frac{\prod_{1\le i<j\le n}(x_{\rho(i)} - x_{\rho(j)})}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{2.8}$$

3. One-dimensional representations

Consider some examples first.

Exercise 3.1. The quaternion group Q_8 is generated by the elements i, j, ε such that $\varepsilon^2 = e$, $\varepsilon i = i\varepsilon$, $\varepsilon j = j\varepsilon$, $i^2 = j^2 = \varepsilon$, $ij = \varepsilon ji$. Find all its 1-dimensional complex representations.

Exercise 3.2. The dihedral group D_n is generated by a rotation a and a reflection s such that $a^n = s^2 = e$, $sas = a^{-1}$. Find all 1-dimensional complex representations of the group D_n. Note that the answer depends on the parity of n.

For a more general approach let us recall useful constructions from Group Theory.

Definition 3.3. Let G be a group, for any $a, b \in G$ define the commutator $[a,b]$ by $[a,b] := aba^{-1}b^{-1}$. Define the derived subgroup $[G,G]$ as a subgroup generated by the set of elements $\{[a,b] \mid a, b \in G\}$.

Proposition 3.4. *Let G be a group, then:*
a) $[G,G]$ is a normal subgroup in G;
b) the quotient group $G/[G,G]$ is abelian;
c) the subgroup $[G,G]$ is a minimal subrgroup of G among the normal subgroups H of G with the property that the quotient group G/H is abelian.

Proposition 3.5. *Assume G is a group, $p : G \to G/[G,G]$ is a canonical projection. For every one-dimensional representation $\alpha : G/[G,G] \to \mathbb{F}^*$ we define the one-dimensional representation $\bar{\alpha} : G \to \mathbb{F}^*$ by $\bar{\alpha} = \alpha \circ p$. The map $\alpha \mapsto \bar{\alpha}$ is a bijection between the set of one-dimensional representations of the group $G/[G,G]$ and the set of one-dimensional representations of the group G.*

Exercise 3.6. Show that $[S_n, S_n] = A_n$, where $A_n := sgn^{-1}(1)$.

Exercise 3.7. Find all 1-dimensional complex representations of the group S_n.

Exercise 3.8. Find all 1-dimensional complex representations of the group A_4.

4. Geometric constructions of representations

Exercise 4.1. Consider an equilateral triangle on the plane with the center at the point $(0, 0)$. The symmetric group S_3 acts on this triangle by permutations of its vertices. Show that the corresponding 2-dimensional representation T of the group S_3 over $\mathbb{R}$ is irreducible. Now let us consider those real-valued matrices $T(\sigma)$, $\sigma \in S_3$ as complex-valued. Show that the corresponding complex representation $T^{\mathbb{C}}$ (complexification) of the group S_3 over $\mathbb{C}$ is also irreducible.

Exercise 4.2. Consider a regular tetrahedron in the Euclidean space $\mathbb{R}^3$ with the center at the origin. Denote by $O(T)$ the set of isometries of $\mathbb{R}^3$ preserving the tetrahedron. Also put $SO(T) = \{A \in O(T) \mid \det(A) > 0\}$. Show that

$$O(T) \simeq S_4, \quad SO(T) \simeq A_4. \tag{4.1}$$

Exercise 4.3. Consider a cube in the Euclidean space $\mathbb{R}^3$ with the center at the origin. Denote by $O(C)$ the set of isometries of $\mathbb{R}^3$ preserving the cube. Also put $SO(C) = \{A \in O(C) \mid \det(A) > 0\}$. Show that

$$O(C) \simeq S_4 \times \mathbb{Z}_2, \quad SO(C) \simeq S_4. \tag{4.2}$$

Exercise 4.4. Show that the three-dimensional real representations of the group A_4 determined by the motions of the regular tetrahedron and the cube are irreducible.

So we get also two three-dimensional irreducible real representations of the group S_4. One can show that because their dimensions are odd, the complexifications of those representations are also irreducible.

Exercise 4.5. Is the representation of the group S_4 in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?

Exercise 4.6. Is the representation of the group A_4 in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?

Exercise 4.7. Let $k \in \mathbb{Z}$. Define a 2-dimensional complex representation T_k of the group D_n as

$$T_k(a) = \begin{pmatrix} \cos\frac{2\pi k}{n} & \sin\frac{2\pi k}{n} \\ -\sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{pmatrix} \qquad T_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Check that the representation T_k is well-defined for all integer k. When is the representation T_k irreducible?

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Section 1.2, 1.3;
- James, Liebeck, Chapters 7,9;
- Vinberg, Sections 1.3, 1.4.