LECTURE 3. MASCHKE'S THEOREM. TENSOR PRODUCT OF REPRESENTATIONS.

Please consider every proposition in this text to be an exercise.

1. Complete reducibility of representations.

Definition 1.1. Let $\mathbb{F}$ be a field. Then we denote by $\mathrm{char}\mathbb{F}$ (the characteristic of the field $\mathbb{F}$) the minimal natural number n such that

$$\underbrace{1+1+\cdots+1}_{n \text{ times}} = 0. \tag{1.1}$$

If there is no such natural number we will write $\mathrm{char}\mathbb{F} := 0$.

Proposition 1.2. *(Maschke) Suppose that a group G is finite, $\mathbb{F}$ is a field, V is a vector space over the field $\mathbb{F}$, $T : G \to \mathrm{GL}(V)$ is a representation, and* $\mathrm{char}\mathbb{F}$ *does not divide $|G|$. Then the representation T is completely reducible.*

Proof. Let $U \subset V$ be an invariant subspace. Denote by $W \subset V$ an arbitrary subspace of V such that $U \oplus W = V$. Define $P \in \mathrm{End}(V)$ by an equality

$$P(u) := u, \quad \forall u \in U, \quad P(w) := 0, \quad \forall w \in W. \tag{1.2}$$

Now we will average P over the group G. Let

$$P_0 := \frac{1}{|G|} \sum_{g \in G} T(g) P T(g^{-1}). \tag{1.3}$$

Exercise 1.3. Show that $P_0 \in \mathrm{End}(T)$.

Thus, $\ker(P_0)$ is invariant wrt T.

Exercise 1.4. Show that

$$U \oplus \ker P_0 = V. \tag{1.4}$$

□

Question 1.5. Suppose that $T : G \to \mathrm{GL}(V)$ is a completely reducible representation and V is expressed as a sum of minimal invariant subspaces:

$$V = \bigoplus_{i=1}^{m} V_i \tag{1.5}$$

Is such expression unique up to permutation of summands?

Exercise 1.6. Suppose that the group G is finite and $T : G \to \mathrm{GL}(\mathbb{C}^2)$ is a representation. Suppose $\exists g, h \in G$ such that $T(g)T(h) \neq T(h)T(g)$. Can the representation T be not irreducible?

Date: September 29, 2024.

2. Dual Representation.

Definition 2.1. Let V be a vector space over a field $\mathbb{F}$. The vector space $\mathrm{Hom}(V, \mathbb{F})$ is called the dual space (to the vector space V) and is denoted by V^*. The elements of V^* are called linear functionals, or covectors, or linear forms.

Definition 2.2. Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V. Define $f^i \in V^*$ at the basis vectors via

$$f^i(v_j) = \delta^i_j, \quad 1 \leq i, j \leq n, \tag{2.1}$$

where δ^i_j is the Kronecker delta:

$$\delta^i_j = \begin{cases} 1, & \text{if } i \neq j; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

The set $\{f^1, f^2, \ldots, f^n\}$ is called the dual basis (to the basis $\{v_1, \ldots, v_n\}$).

Exercise 2.3. Show that $\{f^1, f^2, \ldots, f^n\}$ is a basis of V^*.

Proposition 2.4. *Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$. Then, there exists an isomorphism of V and $(V^*)^*$ such that it does not depend on the choice of basis in V.*

Exercise 2.5. Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V and C is the change of basis matrix from α to β. Denote by γ and δ the bases of V^* dual to α and β, respectively. Express the change-of-basis matrix from γ to δ using the matrix C.

Definition 2.6. Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in Hom(V, U)$. Define $A^* \in \mathrm{Hom}(U^*, V^*)$ by

$$(A^*(f))(v) = f(Av), \quad \forall v \in V, \forall f \in U^*. \tag{2.3}$$

A^* is called the operator adjoint of A.

Exercise 2.7. Show that for any $A \in \mathrm{Hom}(V, U)$ the map $A^* \in \mathrm{Hom}(U^*, V^*)$ exists and it is unique.

Exercise 2.8. Suppose that V and U are finite-dimensional vector spaces, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, $\gamma = \{f^1, \ldots, f^n\}$ is its dual basis, $\beta = \{u_1, \ldots, u_m\}$ is a basis of U, $\delta = \{g^1, \ldots, g^m\}$ is its dual basis, $A \in Hom(V, U)$ and its matrix $[A]^\beta_\alpha$ equals

$$(a^j_i)_{1 \leq i \leq n, 1 \leq j \leq m} \tag{2.4}$$

Find the matrix of $[A^*]^\gamma_\delta$.

Definition 2.9. Let $T : G \to \mathrm{GL}(V)$ be a representation over a field F. Define its dual representation $T^* : G \to \mathrm{GL}(V^*)$ by

$$\boxed{T^*(g) = (T(g^{-1}))^*.} \tag{2.5}$$

Exercise 2.10. Show that T^* is indeed a representation of the group G.

Exercise 2.11. Define $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^3)$ as

$$T(n) = \begin{pmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that T is a representation. Find the matrix of $T^*(n)$ in the dual basis.

Question 2.12. Let $T : G \to \mathrm{GL}(V)$ be a representation. Is it always true that the representations T and $(T^*)^*$ are isomorphic?

Exercise 2.13. Let T be an irreducible representation of a group G. Is it possible that the representation T^* is not irreducible?

3. Tensor Product of Representations.

Definition 3.1. Suppose that V and U are vector spaces over a field $\mathbb{F}$. Denote by $Free(V,U)$ the set of all formal finite linear combinations elements of $V \times U$:

$$Free(V,U) := \left\{ \sum_{i=1}^{n} a_i(x_i, y_i) \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\}. \tag{3.1}$$

(Note that the pairs (x_i, y_i) are simply symbols here. Probably it would be less confusing if we wrote $Free(V,U)$ as

$$\left\{ \sum_{i=1}^{n} a_i e_{(x_i,y_i)} \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\} \tag{3.2}$$

but then we would get more cumbersome expressions further in this definition.)

The set $Free(V,U)$ has the natural structure of a vector space. Now we may define a subspace $Ident(V,U) \subset Free(V,U)$ as

$$\begin{aligned} Ident(V,U) = \mathrm{span}\langle \{(ax+bz,y) - a(x,y) - b(z,y) \,|\, x,z \in V, y \in U, a,b \in \mathbb{F}\} \cup \\ \{(x,ay+bt) - a(x,y) - b(x,t) \,|\, x \in V, y,t \in U, a,b \in \mathbb{F}\} \rangle. \end{aligned} \tag{3.3}$$

The tensor product of vector spaces V and U is, by definition, the quotient space $Free(V,U)$ by $Ident(V,U)$. We denote the tensor product of vector spaces V and U by $V \otimes U$. Also we denote by

$$\sum_{i=1}^{n} a_i x_i \otimes y_i \tag{3.4}$$

the equivalence class that contains the element $\sum_{i=1}^{n} a_i(x_i, y_i)$. A map $\mathrm{emb} : V \times U \to V \otimes U$ is defined by $\mathrm{emb}((v,u)) := v \otimes u$. The elements of $V \otimes U$ are often called tensors.

For example, the element $(a \cdot x + 0 \cdot z, y) - a(x,y) - 0 \cdot (z,y) \in Ident(V,U)$. Therefore, $(ax) \otimes y = a(x \otimes y)$, $\forall a \in \mathbb{F}$. In particular, $0 \otimes y = 0$.

The following statement may be considered a property characterizing the tensor product of two vector spaces.

Proposition 3.2. *Suppose that V, U and W are vector spaces over a field $\mathbb{F}$, and a map $f : V \times U \to W$ satisfies the following conditions:*

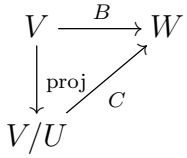
$$\begin{aligned} f(ax+bz,y) &= af(x,y) + bf(z,y), \\ f(x,ay+bt) &= af(x,y) + bf(x,t), \quad \forall x,z \in V, \forall y,t \in U, \forall a,b \in \mathbb{F}. \end{aligned} \tag{3.5}$$

Then there exists a unique map $A \in \mathrm{Hom}(V \otimes U, W)$ such that $A \circ \mathrm{emb} = f$.

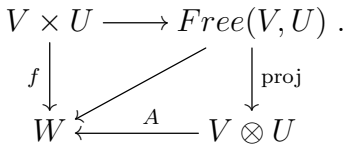
$$\begin{array}{ccc} V \times U & \xrightarrow{\mathrm{emb}} & V \otimes U \\ \downarrow f & \swarrow A & \\ W & & \end{array}$$

To prove Proposition 3.2, we will establish the following lemma first.

Lemma 3.3. *Suppose that V, W are vector spaces, U is a subspace of V, by* $\mathrm{proj} : V \to V/U$ *we denote the canonical projection, and $B \in \mathrm{Hom}(V, W)$. Then, the existence of a map $C \in \mathrm{Hom}(V/U, W)$ such that $C \circ \mathrm{proj} = B$ is equivalent to the condition $U \subseteq \ker(B)$.*



Returning to the proof of Proposition 3.2, we note that the diagram defines the map $g : Free(V, U) \to W$. Then, we may apply Lemma 3.3 to the lower left triangle of the diagram:



Proposition 3.4. *Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V, $\{u_1, \ldots, u_m\}$ is a basis of U. Then*

$$\{v_i \otimes u_j \,|\, 1 \le i \le n, 1 \le j \le m\} \tag{3.6}$$

is a basis of $V \otimes U$.

One may use the result of Proposition 3.2 in the proof of linear independence of the set in (3.6).

Proposition 3.5. *Let V and U be finite-dimensional vector spaces. Define $\varphi : V^* \otimes U \to \mathrm{Hom}(V, U)$ by*

$$\varphi(f \otimes u)(v) := f(v)u, \quad \forall v \in V, u \in U, f \in V^*. \tag{3.7}$$

The map φ is well-defined and $\varphi \in Iso(V^* \otimes U, \mathrm{Hom}(V, U))$.

Note that in the Proposition 3.5, we have defined the map only on the tensors of the form $v \otimes u$. That is not uncommon. One may say that we follow here Proposition 3.2.

As a special case of Proposition 3.5, we may obtain an important corollary.

Proposition 3.6. *Let V be a finite-dimensional vector space. Then the vector spaces $V \otimes V^*$ and $\mathrm{End}(V)$ are canonically isomorphic.*

Definition 3.7. Suppose that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(U)$. Define the tensor product of linear operators $A \otimes B \in \mathrm{End}(V \otimes U)$ by the formula

$$(A \otimes B)(v \otimes u) := (Av) \otimes (Bu) \tag{3.8}$$

on the tensors of the form $v \otimes u$ and then extend by linearity. By definition, we state that

$$A^{\otimes k} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}} \tag{3.9}$$

Exercise 3.8. Show that $A \otimes B$ is well-defined.

Exercise 3.9. Suppose that $A, B \in \mathrm{End}(V)$. Is it always true that

$$(AB)^{\otimes k} = ((A)^{\otimes k})((B)^{\otimes k})? \tag{3.10}$$

Exercise 3.10. Suppose that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \dots, v_n\}$ is a basis of V, $\{u_1, \dots, u_m\}$ is a basis of U, $A \in \mathrm{End}(V)$, its matrix in the basis $\{v_1, \dots, v_n\}$ is

$$(a_i^j)_{1 \le i \le n, 1 \le j \le n}, \tag{3.11}$$

$B \in \mathrm{End}(U)$, its matrix in the basis and $\{u_1, \dots, u_m\}$ is

$$(b_i^j)_{1 \le i \le m, 1 \le j \le m}. \tag{3.12}$$

Find the matrix of $A \otimes B$ in the basis

$$\{v_1 \otimes u_1, v_1 \otimes u_2, \dots, v_1 \otimes u_m, v_2 \otimes u_1, \dots, v_n \otimes u_m\}. \tag{3.13}$$

Question 3.11. Is it always true that

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)? \tag{3.14}$$

Definition 3.12. Let $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ be representations over a field $\mathbb{F}$. We define the tensor product of representations T and S $T \otimes S : G \to \mathrm{GL}(V \otimes U)$ by the following equlaity:

$$\boxed{(T \otimes S)(g) := (T(g)) \otimes (S(g)).} \tag{3.15}$$

Exercise 3.13. Let $T_1, \dots, T_n$ be the set of all pairwise non-isomorphic one-dimensional complex representations of G. Show that this set forms a group, where the tensor product stands for the multiplication, and the inverse element is the dual representation. Describe that group for $G = \mathbb{Z}_n$, S_n.

Exercise 3.14. Let T be a representation of a finite group G. Is it true that the representation $T \otimes T^*$ of the group G always has a one-dimensional invariant subspace ?

4. Symmetric and Exterior Powers of Representations.

Definition 4.1. Let V be a vector space over $\mathbb{F}$. Define the subspaces of k-th symmetric and exterior powers of V, respectively, $S^k(V)$ and $\Lambda^k(V)$ of

$$V^{\otimes k} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}} \tag{4.1}$$

as

$$S^k(V) := \mathrm{span}\langle \sum_{\rho \in S_k} x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \dots, x_k \in V \rangle, \tag{4.2}$$

$$\Lambda^k(V) := \mathrm{span}\langle \sum_{\rho \in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \dots, x_k \in V \rangle. \tag{4.3}$$

We use the notation

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k := \frac{1}{k!} \sum_{\rho \in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)}. \tag{4.4}$$

Exercise 4.2. Suppose that $\dim V = n$. Find $\dim S^k(V)$ and $\dim \Lambda^k(V)$.

Definition 4.3. Suppose that $A \in \mathrm{End}(V)$. One may show that

$$\forall z \in S^k(V) \text{ we get that } A^{\otimes k}(z) \in S^k(V), \tag{4.5}$$

$$\forall z \in \Lambda^k(V) \text{ we get that } A^{\otimes k}(z) \in \Lambda^k(V). \tag{4.6}$$

We are going to denote by $S^k(A) \in \mathrm{End}(S^k(V))$ and $\Lambda^k(A) \in \mathrm{End}(\Lambda^k(V))$ the restrictions of $A^{\otimes k}$ to the subspaces $S^k(V)$ and $\Lambda^k(V)$, respectively.

Definition 4.4. Let $T : G \to \mathrm{GL}(V)$ be a representation. Define the symmetric square $S^2T : G \to \mathrm{GL}(S^2(V))$ and the exterior square $\Lambda^2 T : G \to \mathrm{GL}(\Lambda^2(V)$ of the representation T by

$$\boxed{(S^2T)(g) := S^2(T(g)), \quad (\Lambda^2 T)(g) := \Lambda^2(T(g)).} \tag{4.7}$$

5. Determinant.

Definition 5.1. Suppose that V is a finite-dimensional vector space, $A \in \mathrm{End}(V)$. Note that $\dim(\Lambda^{\dim V}(V)) = 1$. Therefore, $\Lambda^{\dim V}(A)$ is a multiple of the identity map. We will define the determinant $\det(A)$ of the linear operator A by the following equality

$$\boxed{\Lambda^{\dim V}(A) = \det(A) \cdot \mathrm{Id}.} \tag{5.1}$$

Proposition 5.2. *Suppose that* $\dim V < \infty$ *and* $A, B \in \mathrm{End}(V)$. *Then*

$$\det(AB) = \det(A) \cdot \det(B). \tag{5.2}$$

Proposition 5.3. *Suppose that* $\dim V = n < \infty$ *and* $A \in \mathrm{End}(V)$ *has a matrix* $(a_i^j)_{1 \le i,j \le n}$ *in some basis. Then*

$$\boxed{\det(A) = \sum_{\rho \in S_n} \mathrm{sgn}(\rho) \prod_{j=1}^{n} a^j_{\rho(j)}.} \tag{5.3}$$

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Section 1.2;
- James, Liebeck, Chapter 8;
- Vinberg, Sections 1.2, 1.3.