LECTURE 4. ORTHOGONALITY RELATIONS.

Please consider every proposition in this text to be an exercise.

1. Invariant inner product.

First, we revise a couple of facts from Linear Algebra.

Definition 1.1. Assume V is a complex inner product space. An operator $A \in \mathrm{End}(V)$ is called unitary if

$$\langle Ax, Ay\rangle = \langle x, y\rangle, \quad \forall x, y \in V. \tag{1.1}$$

Proposition 1.2. *Assume $A \in \mathrm{End}(V)$ is a unitary operator, a subspace $U \subset V$ is invariant wrt A (i. e. we have that $Au \in U$, $\forall u \in U$). Then $U^{\perp} = \{w \in V \,|\, \langle w, u\rangle = 0, \forall u \in U\}$ is also invariant wrt A.*

Proposition 1.3. *Assume $A \in \mathrm{End}(V)$ is a unitary operator. Then there exist an orthonormal basis $e_1, e_2, \dots, e_n$ of the space V and an n-tuple of complex numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $Ae_i = \lambda_i e_i$. Moreover, $|\lambda_i| = 1$, $\forall 1 \le i \le n$.*

We are ready to give an alternative proof of Maschke's Theorem for the complex case.

Proposition 1.4. *Let G be a finite group and let $T : G \to \mathrm{GL}(V)$ be a complex representation of the group G. Then there exists an inner product in V such that $T(g)$ is unitary for every $g \in G$.*

In other words, we have shown that in the space of a complex representation of a finite group, we can always define an inner product, that is invariant under the action of operators of that representation.

Exercise 1.5. Deduce Maschke's Theorem for the complex case from Proposition 1.4.

2. Character of a representation.

Definition 2.1. Let $T : G \to \mathrm{GL}(V)$ be a representation over a field $\mathbb{F}$. Define the character $\chi_T : G \to \mathbb{F}$ of the representation T by

$$\chi_T(g) := \mathrm{tr}(T(g)), \quad g \in G. \tag{2.1}$$

Proposition 2.2. *a)* $\chi_T(e) = \dim(T)$;
b) $\chi_T(hgh^{-1}) = \chi_T(g)$, $\forall g, h \in G$;
c) χ_T *is constant on the conjugacy classes of the group* G;
d) $\chi_{T\oplus S} = \chi_T + \chi_S$;
e) $\chi_{T\otimes S} = \chi_T \cdot \chi_S$.

Exercise 2.3. Let $T : G \to \mathrm{GL}(V)$ be a complex representation of a finite group G. Show that

$$|\chi_T(g)| \le \dim T, \quad \forall g \in G. \tag{2.2}$$

Exercise 2.4. Assume that $T : G \to \mathrm{GL}(V)$ is a complex representation. Can we express the character $\chi_{\Lambda^2 T}$ using χ_T ?

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3. Decomposition of a representation.

Recall the theorem we proved recently.

Proposition 3.1. *(Maschke's Theorem) Assume that G is a finite group, $T : G \to \mathrm{GL}(V)$ is a representation of the group G over the field $\mathbb{F}$. If $|G|$ is not equal to 0 in the field $\mathbb{F}$ (or, in other words, the characteristic of the field $\mathbb{F}$ does not divide $|G|$) then the representation T is completely reducible.*

Assume $T : G \to \mathrm{GL}(V)$ is a representation of the group G over the field $\mathbb{F}$ of characteristic 0. We have already seen that the decomposition of V into the direct sum of minimal invariant subspaces

$$V = \bigoplus_{i=1}^{k} V_i \tag{3.1}$$

is not uniquely determined up to permutations of summands. But we can establish the uniqueness property for the corresponding subrepresentations

$$T \cong \bigoplus_{i=1}^{k} T|_{V_i}. \tag{3.2}$$

Proposition 3.2. *Assume $T : G \to \mathrm{GL}(V)$ is a representation of the group G and*

$$T \cong \bigoplus_{i=1}^{k} T_i \cong \bigoplus_{j=1}^{m} S_j, \tag{3.3}$$

where $T_1, \dots, T_k, S_1, \dots, S_m$ are irreducible representations. Then $k = m$ and $\exists \sigma \in S_k$ such that $T_{\sigma(i)} \simeq S_i$, $\forall 1 \le i \le k$.

Definition 3.3. Let G be a finite group. Denote by $\widehat{G}$ a set of equivalence classes of irreducible complex representations of the group G. Allowing ourselves a slight ambiguity, we will also denote by $\widehat{G}$ an arbitrary set that contains one representative from each equivalence class of irreducible complex representations of the group G.

Definition 3.4. For a representation $T : G \to \mathrm{GL}(V)$ and a natural number m we will denote:

$$mT := \underbrace{T \oplus T \oplus \cdots \oplus T}_{n \text{ times}}. \tag{3.4}$$

From Maschke's Theorem, we deduce that every real or complex representation T of a finite group G is isomorphic to the sum of irreducible representations with some multiplicities:

$$T \cong \bigoplus_{S \in \widehat{G}} m_s S \tag{3.5}$$

By Proposition 3.2, those multiplicities are uniquely determined by T.

Taking the character of both sides, we get the equality

$$\chi_T = \sum_{S \in \widehat{G}} m_S \chi_S. \tag{3.6}$$

4. Orthogonality relations.

Definition 4.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the set of functions from G to $\mathbb{F}$. The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. For the case $\mathbb{F} = \mathbb{C}$ we introduce an inner product on $\mathbb{C}[G]$ via the formula

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{x \in G} f_1(x)\overline{f_2(x)}. \tag{4.1}$$

Also, define

$$Z(\mathbb{C}[G]) := \{f \in \mathbb{C}[G] \,|\, f(gxg^{-1}) = f(x), \forall x, g \in G\} \tag{4.2}$$

The next lecture will be devoted to the proof of the following remarkable result.

Proposition 4.2. *Let G be a finite group. The set $\{\chi_T \,|\, T \in \widehat{G}\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

There are several corollaries of that claim that we are going to use.

Proposition 4.3. *Suppose that T and S are irreducible complex representations of a finite group G. Then*

$$\langle \chi_T, \chi_S \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_T(g)\overline{\chi_S(g)} = \begin{cases} 0, & \text{if } T \text{ is not isomorphic to } S, \\ 1, & \text{otherwise.} \end{cases} \tag{4.3}$$

Proposition 4.4. *Suppose that T and S are irreducible complex representations of a finite group G. If $\chi_T = \chi_S$, then the representations T and S are isomorphic.*

Proposition 4.5. *Let T be a complex representation of a finite group G. If $\langle \chi_T, \chi_T \rangle = 1$ then T is irreducible.*

Proposition 4.6. *Assume that G is a finite group, $x, y \in G$. Then*

$$\sum_{T \in \widehat{G}} \chi_T(x)\overline{\chi_T(y)} = \begin{cases} \frac{|G|}{|C|}, & \text{if } x \text{ and } y \text{ belong to the same conjugacy class } C, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Proposition 4.7. *Let G be a finite group. Then the following identity holds*

$$\sum_{T \in \widehat{G}} (\dim T)^2 = |G|. \tag{4.5}$$

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 2.1, 2.2;
- James, Liebeck, Chapters 10-16;
- Vinberg, Sections 1.4, 2.5, 2.6.