LECTURE 5. GROUP ALGEBRA.

Please consider every proposition in this text to be an exercise.

1. Group Algebra.

Definition 1.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the following set of formal linear combinations

$$\mathbb{F}[G] = \{\sum_{g\in C} a_g g \,|\, a_g \in \mathbb{F}, \forall g \in G\}. \tag{1.1}$$

The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. Using distributivity, we can extend the operation of multiplication from the group G to $\mathbb{F}[G]$. $\mathbb{F}[G]$ is called a group algebra. It is indeed an unital associative algebra over the field $\mathbb{F}$.

Question 1.2. As a vector space, the group algebra is isomorphic to the space of functions on the group introduced in the previous lecture. How can we interpret the multiplication in the group algebra in terms of functions?

Question 1.3. How can we describe the subspace of central functions in terms of the group algebra?

It is natural to construct an extension of a representation of a group to its group algebra.

Exercise 1.4. For a representation $T : G \to \mathrm{GL}(V)$ of a finite group G over a field $\mathbb{F}$ we can define a unique linear extension of the representation T to the group algebra $\mathbb{F}[G]$, denote it by $\overline{T}$. By definition, $\overline{T}$ maps $\mathbb{F}[G]$ to $\mathrm{End}(V)$. Show that if $\overline{T}(\mathbb{F}[G]) = \mathrm{End}(V)$ then T is irreducible.

2. Orthogonality of matrix elements.

Recall orthogonality relations for irreducible characters.

Proposition 2.1. *Suppose that G is a finite group, T and S are irreducible complex representations of G. Then*

$$\langle \chi_T, \chi_S\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_T(g)\overline{\chi_S(g)} = \begin{cases} 0, & \text{if } T \text{ is not isomorphic to } S, \\ 1, & \text{otherwise.} \end{cases} \tag{2.1}$$

We are going to prove a stronger claim.

Proposition 2.2 (Orthogonality of matrix elements)**.** *Suppose that G is a finite group, $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(W)$ are irreducible complex representations of G, $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$ are invariant inner products in the vector spaces V and W respectively, $x, y \in V$, $z, t \in W$. Then*

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V \overline{\langle S(g)z, t\rangle_W} =$$

$$= \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ \frac{1}{\dim T}\langle x, z\rangle_V \overline{\langle y, t\rangle_V}, & \text{otherwise (here we identify } T \text{ with } S, \text{ and } V \text{ with } W). \end{cases} \tag{2.2}$$

Proof. We are going to use the following lemma from Linear Algebra.

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Lemma 2.3. *Suppose V and W are complex inner product vector spaces, $f : V \times W \to \mathbb{C}$ is a sesquilinear function, i.e. f is linear with respect to the first variable and conjugate linear with respect to the second variable. Then $\exists A \in \mathrm{Hom}(V, W)$ such that*

$$f(x, z) = \langle Ax, z\rangle_W. \tag{2.3}$$

For any $y \in V, t \in W$ put

$$f_1(x, z) = \frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V\overline{\langle S(g)z, t\rangle_W} \tag{2.4}$$

Applying Lemma 2.3 we get that $\exists A \in \mathrm{Hom}(V, W)$ such that

$$f_1(x, z) = \langle Ax, z\rangle_W. \tag{2.5}$$

Then $\forall h \in G$, $\forall x \in V$, $\forall z \in W$ we obtain

$$\begin{aligned}\langle AT(h)x, z\rangle_W = f_1(T(h)x, z) = \frac{1}{|G|}\sum_{g\in G}\langle T(g)T(h)x, y\rangle_V\overline{\langle S(g)z, t\rangle_W} = \\ = \frac{1}{|G|}\sum_{g'\in G}\langle T(g')x, y\rangle_V\overline{\langle S(g')S(h^{-1})z, t\rangle_W} = f_1(x, S(h^{-1})z) = \\ = \langle Ax, S(h^{-1})z\rangle_W = \langle S(h)Ax, z\rangle_W \end{aligned} \tag{2.6}$$

Thus,

$$AT(h) = S(h)A, \quad \forall h \in G. \tag{2.7}$$

Therefore, by Schur's lemma,

$$A = \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ \lambda \cdot Id, & \text{otherwise.} \end{cases} \tag{2.8}$$

Note that the linear map A and, hence, the number λ depend conjugate linearly on y and linearly on t. As a result, we obtain that

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V\overline{\langle S(g)z, t\rangle_W} = \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ f_2(t, y)\langle x, z\rangle_V, & \text{otherwise,} \end{cases} \tag{2.9}$$

where f_2 is a sesquilinear function. Proposition 2.2 has been proven when T is not equivalent to S.

Now let us consider the case when $T = S$, $V = W$. From the equality (2.9) and applying Lemma 2.3 once again we obtain that

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V\overline{\langle T(g)z, t\rangle_V} = c_T\langle x, z\rangle_V\overline{\langle y, t\rangle_V}, \tag{2.10}$$

where c_T is a complex number that depends on T.

Let us find c_T. Suppose that $e_1, e_2, \ldots, e_{\dim T}$ is an orthonormal basis of V. We know that $T(g)$ is a unitary operator for $\forall g \in G$. Therefore,

$$\sum_{i=1}^{\dim T}|\langle T(g)e_1, e_i\rangle|^2 = 1 \tag{2.11}$$

Let us put $x = z = e_1$, $y = t = e_i$ for $1 \le i \le \dim T$ in the equality (2.10) and sum those $\dim T$ equalities. Then using (2.11) we obtain that $\dim T \cdot c_T = 1$. The proposition has been proven. $\square$

Exercise 2.4. Show that Proposition 2.1 follows from Proposition 2.2.

From Proposition 2.1 we obtain that irreducible characters form a set of pairwise orthogonal vectors in the finite-dimensional vector space $Z(\mathbb{C}[G]$. Therefore, we have proven that if the group G is finite then the set $\widehat{G}$ is also finite.

3. Regular representations.

Let us introduce three important representations of an arbitrary finite group G.

Definition 3.1. Let G be a finite group. Define the representations $LReg : G \to \mathrm{GL}(\mathbb{C}[G])$ and $RReg : G \to \mathrm{GL}(\mathbb{C}[G])$ by the following formulas:

$$(LReg(g)f)(x) := f(g^{-1}x), \quad \forall g, x \in G, \forall f \in \mathbb{C}[G]; \tag{3.1}$$

$$(RReg(g)f)(x) := f(xg), \quad \forall g, x \in G, \forall f \in \mathbb{C}[G]. \tag{3.2}$$

Also define the representation $BiReg : G \times G \to \mathrm{GL}(\mathbb{C}[G])$ by

$$(BiReg(g,h)f)(x) := f(g^{-1}xh), \quad \forall g, h, x \in G, \forall f \in \mathbb{C}[G]; \tag{3.3}$$

These representations are called the left regular, the right regular, and the biregular representation respectively.

Exercise 3.2. Find the characters of those three representations.

Exercise 3.3. Find the multiplicities in the decomposition of the representation $LReg$ into the sum of irreducible complex representations of the group G.

4. Algebraic isomorphism of a complex group algebra.

Recall the main statement formulated in the previous lecture.

Proposition 4.1. *Let G be a finite group. The set $\{\chi_T \,|\, T \in \widehat{G}\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

We are going to deduce it from the following beautiful result.

Proposition 4.2. *Suppose that G is a finite group. Then the set $\widehat{G}$ is finite. Denote $k = |\widehat{G}|$ and $\widehat{G} = \{T_i : G \to \mathrm{GL}(V_i) \,|\, 1 \le i \le k\}$. Put*

$$V = \bigoplus_{i=1}^{k} V_i \tag{4.1}$$

Define $B \in \mathrm{Hom}(\mathbb{C}[G], \mathrm{End}(V))$ by

$$B(\sum_{g\in G} a_g g)(v_1, v_2, \ldots, v_k) = (\sum_{g\in G} a_g(T_1(g)(v_1)), \sum_{g\in G} a_g(T_2(g)(v_2)), \ldots, \sum_{g\in G} a_g(T_k(g)(v_k))) \tag{4.2}$$

Then B is an isomorphism of $\mathbb{C}[G]$ and $\prod_{i=1}^{k} \mathrm{End}(V_i)$.

Proof. We need to show that the map B is injective and surjective.

First, let us prove the surjectivity. Consider an arbitrary sequence $D = (D_1, \ldots, D_k), \quad D_i \in \mathrm{End}(V_i)$. It is sufficient to show that we may find a preimage of the sequence D when

$$D_t = 0, \, \forall t \neq s \text{ and } D_s = E_i^j \text{ where } 1 \le s \le k, \, 1 \le i, j \le \dim T_s. \tag{4.3}$$

Here E_i^j denotes the matrix unit with a 1 in the i-th row and the j-th column. WLOG we may assume that $s = 1$. Denote by $\langle \cdot, \cdot \rangle_{V_1}$ an invariant inner product in the vector space V_1. Let

$e_1, \ldots, e_{\dim T_1}$ be the orthonormal basis in which we consider E_i^j. The matrix unit E_i^j is determined by the condition

$$\langle E_i^j e_k, e_l \rangle_{V_1} = \begin{cases} 1, & \text{if } k = j,\ i = l, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Put

$$x_{i,j} := \frac{\dim T_1}{|G|} \sum_{g \in G} \overline{\langle T_1(g) e_j, e_i \rangle_{V_1}}\, g. \tag{4.5}$$

From Proposition 2.2 we get that if $m \neq 1$ then

$$\langle T_m(x_{i,j})u, w \rangle_{V_s} = \frac{\dim T_1}{|G|} \sum_{g \in G} \langle T_m(g)u, w \rangle_{V_m} \overline{\langle T_1(g)e_j, e_i \rangle_{V_1}} = 0, \forall u, w \in V_m. \tag{4.6}$$

Therefore, $T_m(x_{i,j}) = 0$.

Also from Proposition 2.2 we obtain that

$$\langle T_1(x_{i,j})e_k, e_l \rangle_{V_1} = \frac{\dim T_1}{|G|} \sum_{g \in G} \langle T_1(g)e_k, e_l \rangle_{V_1} \overline{\langle T_1(g)e_j, e_i \rangle_{V_1}} = \begin{cases} 1, & \text{if } k = j,\ i = l, \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

Thus, $B(x_{i,j}) = D$ and we have proven that B is surjective.

Exercise 4.3. Show that the map B is injective. Hint: You may consider a regular representation.

□

Exercise 4.4. Describe $B(Z(\mathbb{C}[G]))$.

Exercise 4.5. Suppose that the character of the irreducible representation T_1 is known. Denote by $P \in \mathrm{End}(V)$ a projection operator onto V_1, where

$$V = \bigoplus_{i=1}^{k} V_i \tag{4.8}$$

in the setup of Proposition 4.2. Find a formula for $B^{-1}(P)$.

Exercise 4.6. Deduce Proposition 4.1 from Proposition 4.2.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 2.2 - 2.4;
- James, Liebeck, Sections 10 -16.
- Vinberg, Sections 5, 6.