LECTURE 6. ALGEBRAIC ISOMORPHISM OF A COMPLEX GROUP ALGEBRA. COMPLEXIFICATION OF A REAL REPRESENTATION.

Please consider every proposition in this text to be an exercise.

1. Algebraic isomorphism of a complex group algebra.

In the previous lecture we have shown the following handy formula.

Proposition 1.1 (Orthogonality of matrix elements)**.** *Suppose that G is a finite group, $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(W)$ are irreducible complex representations of G, $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$ are invariant inner products in the vector spaces V and W respectively, $x, y \in V$, $z, t \in W$. Then*

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V\overline{\langle S(g)z, t\rangle_W} = \\ = \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ \frac{1}{\dim T}\langle x, z\rangle_V\overline{\langle y, t\rangle_V}, & \text{otherwise (here we identify } T \text{ with } S, \text{ and } V \text{ with } W). \end{cases} \tag{1.1}$$

Now we are ready to approach the main result formulated in Lecture 4.

Proposition 1.2. *Let G be a finite group. The set $\{\chi_T \,|\, T \in \widehat{G}\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

We are going to deduce it from the following beautiful construction.

Proposition 1.3. *Suppose that G is a finite group. Then the set $\widehat{G}$ is finite. Denote $k = |\widehat{G}|$ and $\widehat{G} = \{T_i : G \to \mathrm{GL}(V_i) \,|\, 1 \le i \le k\}$. Put*

$$V = \bigoplus_{i=1}^{k} V_i \tag{1.2}$$

Define $B \in \mathrm{Hom}(\mathbb{C}[G], \mathrm{End}(V))$ by

$$B(\sum_{g\in G} a_g g)(v_1, v_2, \dots, v_k) = (\sum_{g\in G} a_g(T_1(g)(v_1)), \sum_{g\in G} a_g(T_2(g)(v_2)), \dots, \sum_{g\in G} a_g(T_k(g)(v_k))) \tag{1.3}$$

Then B is an isomorphism of $\mathbb{C}[G]$ and $\prod_{i=1}^{k}\mathrm{End}(V_i)$.

Proof. We need to show that the map B is injective and surjective.

First, let us prove the surjectivity. Consider an arbitrary sequence $D = (D_1, \dots, D_k)$, $\quad D_i \in \mathrm{End}(V_i)$. It is sufficient to show that we may find a preimage of the sequence D when

$$D_t = 0, \forall t \ne s \text{ and } D_s = E_i^j \text{ where } 1 \le s \le k,\ 1 \le i, j \le \dim T_s. \tag{1.4}$$

Here E_i^j denotes the matrix unit with a 1 in the i-th row and the j-th column. WLOG we may assume that $s = 1$. Denote by $\langle\cdot,\cdot\rangle_{V_1}$ an invariant inner product in the vector space V_1. Let $e_1, \dots, e_{\dim T_1}$ be the orthonormal basis in which we consider E_i^j. The matrix unit E_i^j is determined by the condition

$$\langle E_i^j e_k, e_l\rangle_{V_1} = \begin{cases} 1, & \text{if } k = j,\ i = l, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

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Put

$$x_{i,j} := \frac{\dim T_1}{|G|} \sum_{g \in G} \overline{\langle T_1(g)e_j, e_i \rangle_{V_1}}\, g. \tag{1.6}$$

From Proposition 1.1 we get that if $m \neq 1$ then

$$\langle T_m(x_{i,j})u, w \rangle_{V_s} = \frac{\dim T_1}{|G|} \sum_{g \in G} \langle T_m(g)u, w \rangle_{V_m} \overline{\langle T_1(g)e_j, e_i \rangle_{V_1}} = 0, \forall u, w \in V_m. \tag{1.7}$$

Therefore, $T_m(x_{i,j}) = 0$.

Also from Proposition 1.1 we obtain that

$$\langle T_1(x_{i,j})e_k, e_l \rangle_{V_1} = \frac{\dim T_1}{|G|} \sum_{g \in G} \langle T_1(g)e_k, e_l \rangle_{V_1} \overline{\langle T_1(g)e_j, e_i \rangle_{V_1}} = \begin{cases} 1, & \text{if } k = j,\ i = l, \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

Thus, $B(x_{i,j}) = D$ and we have proven that B is surjective.

Exercise 1.4. Show that the map B is injective. Hint: You may consider a regular representation.

□

Exercise 1.5. Describe $B(Z(\mathbb{C}[G]))$.

Exercise 1.6. Suppose that the character of the irreducible representation T_1 is known. Denote by $P \in \mathrm{End}(V)$ a projection operator onto V_1, where

$$V = \bigoplus_{i=1}^{k} V_i \tag{1.9}$$

in Proposition 1.3. Find a formula for $B^{-1}(P)$.

Exercise 1.7. Deduce Proposition 1.2 from Proposition 1.3.

2. External tensor product of representations

Definition 2.1. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(V)$ are representations over the same field. Define an external tensor product $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ of representations T and S by

$$(T \otimes S)(g, h) := T(g) \otimes S(h), \quad \forall g \in G, \forall h \in H. \tag{2.1}$$

It might be confusing that we are using for the external tensor product the same symbol as for the usual tensor product of representations of the same group.

Proposition 2.2. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(V)$ are representations over the same field. Then the representation $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ is irreducible if and only if both T and S are irreducible.*

Proposition 1.3 implies the following nice corollary.

Proposition 2.3. *Let G be a finite group. Then we have*

$$BiReg \simeq \bigoplus_{T \in \widehat{G}} T \otimes T^*. \tag{2.2}$$

3. Complexification of real vector spaces.

Definition 3.1. Let V be a real vector space. Let us consider the field $\mathbb{C}$ as another vector space over $\mathbb{R}$. The tensor product $\mathbb{C} \otimes V$ is also a real vector space. We will introduce a structure of complex vector space on $\mathbb{C} \otimes V$ by setting

$$\lambda \cdot (z \otimes v) := (\lambda z \otimes v), \quad \forall \lambda, z \in \mathbb{C}, \forall v \in V. \tag{3.1}$$

Here, we define the multiplication by a fixed λ on the set of pairs $\mathbb{C} \times V$. Note that it is bilinear, and then we extend this map to the tensor product $\mathbb{C} \otimes V$ using its universal property. We also make the addition operation the same as in the tensor product $\mathbb{C} \otimes V$. The complex vector space that we have obtained is called the complexification of the real vector space V and is denoted by $V^{\mathbb{C}}$. Also put

$$V_0^{\mathbb{C}} := \{1 \otimes v \mid v \in V\}. \tag{3.2}$$

For every subspace U of the vector space V denote by $U^{\mathbb{C}}$ the following subspace of $V^{\mathbb{C}}$:

$$U^{\mathbb{C}} := \{z \otimes u \mid z \in \mathbb{C}, u \in U\} \tag{3.3}$$

Exercise 3.2. Let V be a real vector space. Show that $\forall v \in V^{\mathbb{C}}$ $\exists u, w \in V$ such that

$$v = 1 \otimes u + i \otimes w. \tag{3.4}$$

Exercise 3.3. Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V < \infty$, then the dimension of $V^{\mathbb{C}}$ over C equals $\dim V$. What can we say about $V_0^{\mathbb{C}}$ and its dimension?

Definition 3.4. Suppose that V is a real vector space and $A \in \mathrm{End}(V)$. We define the complexification $A^{\mathbb{C}} \in \mathrm{End}(V^{\mathbb{C}})$ of this linear operator A by

$$A^{\mathbb{C}}(z \otimes v) := z \otimes (Av). \tag{3.5}$$

Note that

$$\forall v \in V_0^{\mathbb{C}} \quad A^{\mathbb{C}} v \in V_0^{\mathbb{C}}, \tag{3.6}$$

and the restriction of $A^{\mathbb{C}}$ to $V_0^{\mathbb{C}}$ coincides with A after applying a natural isomorphism between $V_0^{\mathbb{C}}$ and V. So we can view $A^{\mathbb{C}}$ as an extension of a copy of A defined on $V_0^{\mathbb{C}}$. We are going to exploit this viewpoint a lot in the next section.

Example 3.5. Suppose that V is a real vector space, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, $A \in \mathrm{End}(V)$. Then the set $\beta = \{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ is a basis of $V^{\mathbb{C}}$ and the corresponding matrices coincide.

Proposition 3.6. *Suppose that V is a real vector space, $\dim V < \infty$. Define $\mathrm{Conj} \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ by*

$$\mathrm{Conj}(z \otimes v) := \overline{z} \otimes v. \tag{3.7}$$

Then:

a) $\mathrm{Conj}^2 = \mathrm{Id}$*;*

b) $\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ A^{\mathbb{C}} = A^{\mathbb{C}} \circ \mathrm{Conj}$*;*

c) We get that $\frac{1}{2}(\mathrm{Id} + \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and it is a projection operator onto* $V_0^{\mathbb{C}}$*;*

d) We get that $\frac{1}{2i}(Id - \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and*

$$\mathrm{im}\left(\frac{1}{2i}(Id - \mathrm{Conj})\right) = V_0^{\mathbb{C}}; \tag{3.8}$$

e)

$$\mathrm{im}\left(\frac{1}{2}(\mathrm{Id} + \mathrm{Conj})\right) \oplus \mathrm{im}\left(\frac{1}{2}(Id - \mathrm{Conj})\right) = (V^{\mathbb{C}})^{\mathbb{R}}. \tag{3.9}$$

Proposition 3.7. *Suppose that V is a real finite-dimensional vector space and W is a subspace of $V^{\mathbb{C}}$. Then the following conditions are equivalent:*

$$(\mathrm{Conj}(W) = W) \quad \Longleftrightarrow \quad (\textit{There exists a subspace } U \subset V \textit{ such that } W = U^{\mathbb{C}}). \tag{3.10}$$

4. Complexification of a real representation.

Definition 4.1. Let $T : G \to \mathrm{GL}(V)$ be a real representation. Define the complex representation $T^{\mathbb{C}} : G \to \mathrm{GL}(V^{\mathbb{C}})$ by

$$T^{\mathbb{C}}(g) := (T(g))^{\mathbb{C}}, \quad \forall g \in G. \tag{4.1}$$

Proposition 4.2. *Let $T : G \to \mathrm{GL}(V)$ be an irreducible real representation and let* $\dim T$ *be odd. Then $T^{\mathbb{C}}$ is an irreducible complex representation.*

Exercise 4.3. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is an irreducible complex representation.

Exercise 4.4. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is not irreducible.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Section 2.4;
- James, Liebeck, Sections 16, 19, 20.
- Vinberg, Sections 3-6.