LECTURE 7. IRREDUCIBLE COMPLEX CHARACTERS OF SYMMETRIC GROUPS.

1. Partitions.

Definition 1.1. We will call a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ a partition (of a number n) if:

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$;
(2) $|\lambda| := \sum_{j=1}^{k} \lambda_j = n$.

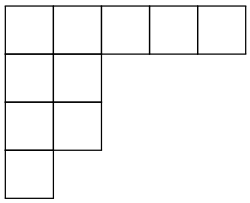
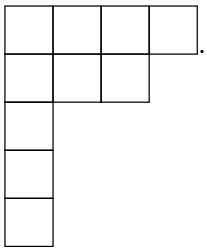
The length of a partition is defined as $l(\lambda) := k$. Denote

$$m_j(\lambda) := \#\{i \,|\, \lambda_i = j\} \tag{1.1}$$

Also we are going to write that $|\lambda| = n$.

With every partition λ we associate a Young diagram D_λ with λ_j boxes in the j-th row, the rows of boxes lined up on the left. The conjugate partition λ' is obtained by interchanging the rows and the columns in the Young diagram D_λ.

For example, consider $\lambda = (5, 2, 2, 1)$. Then $\lambda' = (4, 3, 1, 1, 1)$, and the corresponding Young diagrams look as follows:

$D_\lambda =$  , $D_{\lambda'} =$  . (1.2)

Denote by $p(n)$ the number of partitions of the number n.

Definition 1.2. Assume that ρ is a partition of n. Denote by M_ρ the subset of the group S_n consisting of elements of the cycle type ρ.

Proposition 1.3. *For a partition ρ of n denote*

$$z_\rho = \prod_{j=1}^{l(\rho)} j^{m_j(\rho)}((m_j(\rho))!) \tag{1.3}$$

Show that

$$|M_\rho| = \frac{n!}{z_\rho}. \tag{1.4}$$

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2. Symmetric functions.

Consider two important families of symmetric polynomials.

Definition 2.1. Define Newton symmetric polynomials as

$$p_k(x_1, x_2, \dots, x_m) := \sum_{j=1}^{m} x_j^k; \quad p_\rho(x_1, x_2, \dots, x_m) := \prod_{i=1}^{l(\rho)} p_{\rho_i}. \tag{2.1}$$

Definition 2.2. If $l(\lambda) \leq m$ we define Schur symmetric polynomials as

$$s_\lambda(x_1, x_2, \dots, x_m) := \frac{\det(x_j^{\lambda_i+m-i})_{1\leq i,j\leq m}}{\det(x_j^{m-i})_{1\leq i,j\leq m}} = \frac{\det(x_j^{\lambda_i+m-i})_{1\leq i,j\leq m}}{\prod_{1\leq i<j\leq m}(x_i - x_j)}. \tag{2.2}$$

3. Frobenius formula for irreducible complex characters of symmetric groups.

There is a natural way to parametrize irreducible complex representations of the symmetric group by the partitions of the number n. Let us briefly discuss it.

Proposition 3.1. *Denote by η_n the trivial character of S_n. We will assume that $\eta_k = 0$ if $k < 0$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n and let*

$$S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k} \tag{3.1}$$

be the specific subgroup of S_n. Then

$$\eta_{\lambda_1} \cdot \eta_{\lambda_2} \dots \eta_{\lambda_k} \tag{3.2}$$

is a trivial character of S_λ. Denote by η_λ the character of S_n induced by the trivial character of S_λ. Define

$$\chi^\lambda := \det[\eta_{\lambda_i - i + j}]_{1\leq i,j\leq k}, \tag{3.3}$$

where each summand in the determinant expansion stands for the character that we will induce to the group S_n. Then χ^λ is an irreducible complex character of the group S_n, and we obtain a bijection between the set of partitions of the number n and the set $\widehat{S_n}$.

Definition 3.2. Suppose λ and ρ are partitions of n. Let us introduce the following notation:

$$\alpha^\lambda := \frac{1}{|S_n|} \sum_{g\in S_n} \chi^\lambda(g) g \in Z(\mathbb{C}[S_n]); \quad C_\rho := \frac{1}{|M_\rho|} \sum_{g\in M_\rho} g \in Z(\mathbb{C}[S_n]). \tag{3.4}$$

Also let us denote by χ^λ_ρ the value of the character χ^λ at the element of the cycle type ρ.

Proposition 3.3. *Let λ be a partition of n. Then we have the following equality*

$$\alpha^\lambda = \sum_{|\rho|=n} \chi^\lambda_\rho \frac{C_\rho}{z_\rho}. \tag{3.5}$$

Proposition 3.4. *Let λ be a partition of a number n. Then we have the equality*

$$s_\lambda = \sum_{|\rho|=n} \chi^\lambda_\rho \frac{p_\rho}{z_\rho}. \tag{3.6}$$

Let ρ be a partition of n. Then we get

$$p_\rho = \sum_{|\lambda|=n} \chi^\lambda_\rho s_\lambda. \tag{3.7}$$

Definition 4.1. Consider a partition λ of a number n. A standard tableau of the shape λ is a way to fill in all the boxes of its Young diagram D_λ with the numbers $1, \dots, n$ so that the following conditions hold:

(1) each number is written only in one box;
(2) the numbers go in a strictly increasing order along each row;
(3) the numbers go in a strictly increasing order along each column.

Definition 4.2. Consider a partition λ of a number n. The number of standard tableaux of the shape λ is called the dimension of the partition λ and is denoted by $\dim\lambda$.

Here is an example of the standard tableau of the shape $(5, 2, 2, 1)$:

1	3	5	6	9
2	7			
4	10			
8				

Proposition 4.3. *Assume $|\mu| = n$ then we have*

$$s_\mu \cdot p_{(1)} = \sum_\lambda s_\lambda, \tag{4.1}$$

where the sum runs over the partitions λ such that $|\lambda| = n + 1$ and $D_\mu \subset D_\lambda$.

It implies the following identity for the dimensions of partitions.

Proposition 4.4.

$$p_{(1)}^n = \sum_{\lambda \vdash n} \dim\lambda \cdot s_\lambda. \tag{4.2}$$

That result allows us to obtain a closed formula for the dimension of a partition.

Proposition 4.5. *Let λ be a partition. Then*

$$\dim\lambda = \frac{|\lambda|!}{\prod_{j=1}^{l(\lambda)}(\lambda_j + l(\lambda) - j)!} \prod_{1 \le i < j \le l(\lambda)} (\lambda_i - \lambda_j + j - i). \tag{4.3}$$

Definition 4.6. For a cell $(i, j) \in D_\lambda$ define its hook length as

$$h(i, j) := \lambda_i + \lambda'_j - i - j + 1. \tag{4.4}$$

Proposition 4.7. *(Hook length formula) Let λ be a partition. Then*

$$\dim\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in D_\lambda} h(i, j)}. \tag{4.5}$$

Definition 4.8. Consider a partition λ of a number n and an arbitrary positive integer k. Let us fill in each box of the Young diagram D_λ with one of the numbers $1, \dots, k$, The result is called a semistandard tableau of shape λ (filled with the numbers $1, \dots, k$) if the entries weakly increase along each row and strictly increase down each column. For a semistandard tableau T and some variables $x_1, \dots, x_k$ put

$$x^T := \prod_{j=1}^{k} x_j^{m_j(T)}, \tag{4.6}$$

where $m_j(T)$ is the number of entries in the table T that are equal to j.

We can obtain the following combinatorial description of Schur functions.

Proposition 4.9. *Assume λ is a partition, $k \geq l(\lambda)$, $x_1, \ldots, x_k$ are some variables. Then*

$$s_\lambda(x_1, \ldots, x_k) = \sum_T x^T, \tag{4.7}$$

where the sum is taken over semistandard tableau of shape λ filled with the numbers $1, \ldots, k$.

5. Murnaghan-Nakayama rule.

We can obtain a combinatorial algorithm how to compute χ^λ_ρ.

Definition 5.1. Assume that $D_\mu \subset D_\lambda$. Then the pair (μ, λ) (the difference $D_\lambda \setminus D_\mu$) is called a border strip if the following conditions hold:

(1) $D_\lambda \setminus D_\mu$ has no 2×2 squares;
(2) $D_\lambda \setminus D_\mu$ is connected, where we assume that the path can go either horizontally or vertically.

Denote by $\mathrm{height}(\lambda \setminus \mu)$ the number of rows $D_\lambda \setminus D_\mu$ touches minus one.

Proposition 5.2. *Assume $|\mu| = n$ then we have*

$$s_\mu \cdot p_r = \sum_\lambda (-1)^{\mathrm{height}(\lambda \setminus \mu)} s_\lambda, \tag{5.1}$$

where the sum runs over the partitions λ such that $|\lambda| = n + r$, $D_\mu \subset D_\lambda$ and $D_\lambda \setminus D_\mu$ is a border strip .

Proposition 5.3. *(Murnaghan–Nakayama rule) Assume that $|\lambda| = |\rho| = n$. Then we get*

$$\chi^\lambda_\rho = \sum_T (-1)^{\mathrm{height(T)}}, \tag{5.2}$$

where the sum runs over the sequences of partitions

$$T = (\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \ldots \lambda^{(l(\rho)-1} \subset \lambda^{(l(\rho)} = \lambda) \tag{5.3}$$

such that

(1) *$(\lambda^{(j-1)}, \lambda^{(j)})$ is a border strip for $1 \leq j \leq l(\rho)$;*
(2) *$|\lambda^{(j)}| = |\lambda^{(j-1)}| + \rho_j$ for $1 \leq j \leq l(\rho)$;*

and

$$\mathrm{height(T)} := \sum_{\mathrm{j=1}}^{\mathrm{l}(\rho)} \mathrm{height}(\lambda^{(\mathrm{j})} \setminus \lambda^{(\mathrm{j}-1)}). \tag{5.4}$$