LECTURE 8. MATRIX LIE GROUPS.

Please consider every proposition in this text to be an exercise.

1. Exponent of a matrix

Definition 1.1. For $A \in \mathrm{End}(\mathbb{C}^n)$ define its norm by the formula

$$\|A\| := \sqrt{\sum_{1\le i,j\le n} |a_i^j|^2},$$

where (a_i^j) is the matrix of the operator A in the standard basis.

Proposition 1.2. *The normed space* $\mathrm{End}(\mathbb{C}^n)$ *is complete.*

Proposition 1.3. *For any* $A, B \in \mathrm{End}(\mathbb{C}^n)$ *we have the inequalities*

$$\|A+B\| \le \|A\| + \|B\|; \quad \|A\cdot B\| \le \|A\|\cdot\|B\|. \tag{1.1}$$

Proposition 1.4. *For all* $A \in \mathrm{End}(\mathbb{C}^n)$ *the series*

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

converges absolutely in the space $\mathrm{End}(\mathbb{C}^n)$ *with respect to this norm.*

Exercise 1.5. Compute $\exp(A)$ when A is a diagonal matrix.

Exercise 1.6. Compute $\exp(A)$ when A is a Jordan block.

Proposition 1.7. *For any* $A, B \in \mathrm{End}(\mathbb{C}^n)$ *we have the following properties of the exponent:*
a) $\exp(A)$ *is invertible and* $(\exp(A))^{-1} = \exp(-A)$*;*
b) $(\exp(A))^t = \exp(A^t)$*;*
c) if $AB = BA$ *then* $\exp(A+B) = \exp(A)\exp(B)$*;*
d) If B *is invertible then* $B\exp(A)B^{-1} = \exp(BAB^{-1})$*;*
e) $\exp(sA)$ *is differentiable as a function of* s *and* $\frac{d}{ds}\exp(sA) = A\exp(sA)$.

Exercise 1.8. Suppose that V is a complex finite dimensional vector space and $A \in \mathrm{End(V)}$. Show that $\det(\exp(A)) = e^{\mathrm{tr}(A)}$.

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Exercise 1.9. Assume that $A \in \mathrm{End}(\mathbb{C}^n)$. Then the map $T_A : \mathbb{R} \to \mathrm{End}(\mathbb{C}^n)$ defined by the formula

$$T_A(s) = \exp(sA)$$

is a representation of the additive group of the field $\mathbb{R}$.

a) Describe conditions on A and n equivalent to the condition that T_A is irreducible.

b) Assume

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

What manifold does the set $\{T_A(s) \,|\, s \in \mathbb{R}\}$ form ?

Chapters from the textbooks relevant to the lecture:

- Hall, Sections 2.1, 2.2.