LECTURE 9. MATRIX LIE GROUPS.

Please consider every proposition in this text to be an exercise.

1. Definition and examples

Definition 1.1. Assume that $G \subset GL(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We call G a matrix Lie group if:
1) G is a group with respect to matrix multiplication;
2) G is a closed subset of the topological space $GL(n, \mathbb{F})$.

Consider some examples of matrix Lie groups.
Upper-triangular invertible matrices

$$UT_n(\mathbb{F}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{F}) \mid \det A \neq 0,\ a_i^j = 0,\ \forall 1 \leq i < j \leq n\} \tag{1.1}$$

Unitriangular matrices

$$UUT_n(\mathbb{F}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{F}) \mid \det A \neq 0,\ a_i^j = 0,\ \forall 1 \leq i < j \leq n,\ a_i^i = 1, \forall 1 \leq i \leq n\} \tag{1.2}$$

$UUT_3(\mathbb{C})$ is called the Heisenberg group. General affine group

$$GA_n(\mathbb{F}) = \{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL_n(\mathbb{F}), b \in \mathbb{F}^n\} \tag{1.3}$$

Below are examples of semisimple Lie groups that we will study later. General linear groups over $\mathbb{R}$ and $\mathbb{C}$

$$GL_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \det A \neq 0\} \tag{1.4}$$
$$GL_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid \det A \neq 0\} \tag{1.5}$$

Special linear groups over $\mathbb{R}$ and $\mathbb{C}$

$$SL_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \det A = 1\} \tag{1.6}$$
$$SL_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid \det A = 1\} \tag{1.7}$$

Orthogonal groups over $\mathbb{R}$ and $\mathbb{C}$

$$O_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^t = Id\} \tag{1.8}$$
$$O_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot A^t = Id\} \tag{1.9}$$

Special orthogonal groups over $\mathbb{R}$ and $\mathbb{C}$

$$SO_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^t = Id, \det A = 1\} \tag{1.10}$$
$$SO_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot A^t = Id, \det A = 1\} \tag{1.11}$$

Date: November 10, 2024.

Symplectic groups over $\mathbb{R}$ and $\mathbb{C}$

$$Sp_n(\mathbb{R}) = \{A \in \mathrm{Mat}(2n \times 2n, \mathbb{R}) \mid A \cdot J \cdot A^t = J\} \tag{1.12}$$

$$Sp_n(\mathbb{C}) = \{A \in \mathrm{Mat}(2n \times 2n, \mathbb{C}) \mid A \cdot J \cdot A^t = J\}, \tag{1.13}$$

where

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}. \tag{1.14}$$

Unitary group and special unitary group

$$U_n = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot \bar{A}^t = Id\} \tag{1.15}$$

$$SU_n = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot \bar{A}^t = Id, \det A = 1\} \tag{1.16}$$

Definition 1.2. A matrix Lie group $G \subset GL(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ is called connected if G is path-connected as a subset of $\mathrm{End}(\mathbb{F}^n)$.

A matrix Lie group $G \subset GL(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ is called compact if G is a compact set as a subset of $\mathrm{End}(\mathbb{F}^n)$.

Exercise 1.3. Which of the Lie groups from the list

$$GL_n(\mathbb{R}), GL_n(\mathbb{C}), SL_n(\mathbb{R}), SL_n(\mathbb{C}), SO_n(\mathbb{R}), SO_n(\mathbb{C}), Sp_n(\mathbb{R}), Sp_n(\mathbb{C}), SU_n$$

are compact?

Exercise 1.4. Show that the Lie group $SO_n(\mathbb{R})$ is connected.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 7.1 - 7.3;
- Hall, Sections 1.1 - 1.5;
- Vinberg, Section 10.