LECTURE 11. LIE GROUPS AND LIE ALGEBRAS REPRESENTATIONS.

Please consider every proposition in this text to be an exercise.

1. Irreducible representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra over a field F. A representation of a Lie algebra $\mathfrak{g}$ over the field F is a linear map

$$\phi : \mathfrak{g} \to \mathfrak{gl}(F^n)$$

such that

$$\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X), \quad \forall X, Y \in \mathfrak{g}.$$

Another way to look at it is to say that ϕ is a homomorphism of the Lie algebras $\mathfrak{g}$ and $\mathfrak{gl}(F^n)$.

Question 1.2. Assume that $\mathfrak{g}$ is a Lie algebra over a field F, V is a vector space over that field. Does there always exist a representation of $\mathfrak{g}$ in V?

Exercise 1.3. Suppose $\mathfrak{g}$ is a Lie algebra. Define the map $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by the formula $(\mathrm{ad}_{\mathfrak{g}}(x))(y) := [x, y]$. Show that $\mathrm{ad}_{\mathfrak{g}}$ is a representation of the Lie algebra $\mathfrak{g}$. It is called the adjoint representation of the Lie algebra $\mathfrak{g}$.

The representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ play a special role in Lie theory.
We are going to consider the following basis in $\mathfrak{sl}(2,\mathbb{C})$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.1}$$

They satisfy the following commutator relations:

$$[H, E] = 2E; \quad [H, F] = -2F; \quad [E, F] = H. \tag{1.2}$$

Proposition 1.4. *For every non-negative integer n there exists a unique irreducible $(n+1)$-dimensional representation ρ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We can pick a basis $v_0, v_1, \ldots, v_n$ in the space V of that representation with the following properties:*

$$\rho(H)v_k = (n-2k)v_k, \quad \forall 0 \le k \le n; \tag{1.3}$$
$$\rho(E)v_k = k(n+1-k)v_{k-1}, \quad \forall 1 \le k \le n, \tag{1.4}$$
$$\rho(E)v_0 = 0; \tag{1.5}$$
$$\rho(F)v_k = v_{k+1}, \quad \forall 0 \le k \le n-1, \tag{1.6}$$
$$\rho(F)v_n = 0. \tag{1.7}$$

The number n is called the highest weight of the representation ρ, the vector v_0 is called the highest weight vector, the representation ρ (or, the vector space V with the action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$) is called the highest weight $\mathfrak{sl}_2(\mathbb{C})$-module.

Proof. There exists an eigenvector v of the endomorphism $\rho(h)$:

$$\rho(H)v = \lambda v. \tag{1.8}$$

Question 1.5. Why can we make that claim?

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Exercise 1.6. Show that $\rho(E)v$ is also an eigenvector of $\rho(H)$ with the eigenvalue $\lambda + 2$.

Exercise 1.7. Show that there exist k such that $(\rho(E))^k v = 0$.

Denote by v_0 an eigenvector of $\rho(H)$ such that $\rho(E)v_0 = 0$. Denote by μ the corresponding eigenvalue:

$$\rho(H)v_0 = \mu v_0. \tag{1.9}$$

Also, put

$$v_k := (\rho(F))^k v_0. \tag{1.10}$$

Exercise 1.8. Show that

$$\rho(H)v_k = (\mu - 2k)v_k. \tag{1.11}$$

Exercise 1.9. Show that there exists n such that $(\rho(F))^n v_0 \neq 0$ and $(\rho(F))^{n+1} v_0 = 0$

Denote by a_k the following sequence of complex numbers:

$$\rho(E)v_k = a_k v_{k-1}, \quad 1 \leq k \leq n \tag{1.12}$$

Exercise 1.10. Show that the sequence a_k is well-defined.

Exercise 1.11. Show that

$$a_1 = \mu, \quad a_{k+1} = a_k + (\mu - 2k), \quad 0 \leq k \leq n-1. \tag{1.13}$$

Exercise 1.12. Suggest a formula for a_k.

Exercise 1.13. Show that $\mu = n$.

Therefore, we have proved that if ρ is an irreducible $(n+1)$-dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ then we can pick a basis $v_0, \ldots, v_n$ with the properties described by the equalities 1.3 - 1.7.

Exercise 1.14. Show that the equalities 1.3 - 1.7 define a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

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2. Tangent map of a Lie group representation

Definition 2.1. Suppose that G and H are matrix Lie groups. Then a map $F : G \to H$ is called Lie groups homomorphism if it is a group homomorphism that is continuous.

And, if $H = \mathrm{GL}(V)$ we call that homomorphism a representation of the matrix Lie group G in the vector space V.

We will take for granted that one can show that F is smooth in the sense of the following definition.

Definition 2.2. Assume that G and H are matrix Lie groups and $F : G \to H$ is a Lie group homomorphism. Define the tangent map $dF : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$ by the following rule:

For every $X \in T_{\mathrm{Id}}G$ consider the corresponding to X path $A \in P_{\mathrm{Id}}G$. Then we take for granted that $F(A) \in P_{\mathrm{Id}}H$ and we put

$$dF(X) := \frac{d}{dt}\bigg|_{t=0} (F(A(t))) \tag{2.1}$$

Also, we will take for granted that the tangent map is well-defined.

Proposition 2.3. *Assume that G and H are matrix Lie groups,$F : G \to H$ is a Lie groups homomorphism. Then $dF : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$ is a homomorphism of the Lie algebras, i.e.*

$$dF([X, Y]) = [dF(X), dF(Y)], \quad \forall X, Y \in T_{\mathrm{Id}}G \tag{2.2}$$

Definition 2.4. If $H = \mathrm{GL}(V)$ then the corresponding representation of the Lie algebra of the Lie group G, $dF : T_{\mathrm{Id}}G \to \mathfrak{gl}(V)$, is called the tangent representation.

Exercise 2.5. Let T be an irreducible representation of a matrix Lie group G. Does its tangent representation dT have to be irreducible?

Exercise 2.6. Suppose T is an irreducible representation of a matrix Lie group G and its tangent representation dT is irreducible. Does T have to be irreducible as well?

Consider an important example. Let V be the complex vector space of polynomials $P(x, y)$ in variables x and y. Define the map $T : SL_2(\mathbb{C}) \to GL(V)$ by

$$\left(\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) P\right)(x, y) := P(ax + cy, bx + dy).$$

Denote by $V_n \subset V$ the subspace of homogeneous polynomials of degree n and by T_n the restriction of T on that subspace.

Exercise 2.7. Find the images $dT_n(E), dT_n(H), dT_n(F)$ of the canonical basis elements of $\mathfrak{sl}_2(\mathbb{C})$.

Exercise 2.8. Show that the representation dT_n of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is irreducible.

Let us introduce another widely used example. Denote by Ad : $SU_2 \to \mathrm{GL}(\mathfrak{su}_2)$ the representation defined via $\mathrm{Ad(A)(X)} = \mathrm{AXA}^{-1}$.

Question 2.9. Is Ad a representation of the Lie group SU_2?

Recall that the Lie group SU_2 can be defined as a set of the following matrices

$$SU_2 = \{\begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & a_1 - ia_2 \end{pmatrix} \mid a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1,\ a_1, a_2, a_3, a_4 \in \mathbb{R}\}? \tag{2.3}$$

An arbitrary element of the Lie algebra $\mathfrak{su}_2$ can be represented as

$$\begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}, x_1, x_2, x_3 \in \mathbb{R} \tag{2.4}$$

We get that $\det(X) = x_1^2 + x_2^2 + x_3^2$ is preserved by the map Ad. Thus we get that $\Im(Ad) \subset O(\mathfrak{su}_2) \simeq O_3(\mathbb{R})$.

Exercise 2.10. Show that $\mathrm{im}(Ad) = SO(\mathfrak{su}_2)$.

Exercise 2.11. Find the kernel of this map.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 8.2, 8.3, 11.1;
- Hall, Sections 3.1, 4.6;
- Vinberg, Section 10, 11.