LECTURE 12. LIE GROUPS AND LIE ALGEBRAS REPRESENTATIONS, PART 2.

Please consider every proposition in this text to be an exercise.

1. Tangent map of a Lie group representation

Definition 1.1. Suppose that G and H are matrix Lie groups. Then a map $F : G \to H$ is called Lie groups homomorphism if it is a group homomorphism that is continuous.

If $H = \mathrm{GL}(V)$, we call that homomorphism a representation of the matrix Lie group G in the vector space V.

Definition 1.2. Assume that G and H are matrix Lie groups and $F : G \to H$ is a Lie group homomorphism. Define the tangent map $dF : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$ by the following rule:

For every $X \in T_{\mathrm{Id}}G$ consider the corresponding to X path $A \in P_{\mathrm{Id}}G$. Then we take for granted that $F(A) \in P_{\mathrm{Id}}H$ and we put

$$dF(X) := \frac{d}{dt}\Big|_{t=0} (F(A(t))) \tag{1.1}$$

Also, we will take for granted that the tangent map is well-defined.

Proposition 1.3. *Assume that G and H are matrix Lie groups,$F : G \to H$ is a Lie groups homomorphism. Then $dF : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$ is a homomorphism of the Lie algebras, i.e.*

$$dF([X, Y]) = [dF(X), dF(Y)], \quad \forall X, Y \in T_{\mathrm{Id}}G \tag{1.2}$$

Definition 1.4. If $H = \mathrm{GL}(V)$, then the corresponding representation of the Lie algebra of the Lie group G, $dF : T_{\mathrm{Id}}G \to \mathfrak{gl}(V)$, is called the tangent representation.

Exercise 1.5. Let T be an irreducible representation of a matrix Lie group G. Does its tangent representation dT have to be irreducible?

Exercise 1.6. Suppose T is an irreducible representation of a matrix Lie group G and its tangent representation dT is irreducible. Does T have to be irreducible as well?

2. Representations of the Lie group $SL_2(\mathbb{C})$ in spaces of polynomials

Consider an important example. Let V be the complex vector space of polynomials $P(x, y)$ in variables x and y. Define the map $T : SL_2(\mathbb{C}) \to GL(V)$ by

$$\left(\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) P\right)(x, y) := P(ax + cy, bx + dy).$$

Denote by $V_n \subset V$ the subspace of homogeneous polynomials of degree n and by T_n the restriction of T on that subspace.

Exercise 2.1. Show that T_n is a representation of the Lie group $SL_2(\mathbb{C})$

Exercise 2.2. Find the images $dT_n(E), dT_n(H), dT_n(F)$ of the canonical basis elements of $\mathfrak{sl}_2(\mathbb{C})$.

Exercise 2.3. Show that the representation dT_n of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is irreducible.

Date: December 1, 2024.

Exercise 2.4. Find an explicit isomorphism between the representation dT_n and the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module with the highest weight n by describing the preimages of the basis vectors $v_0, v_1, \ldots, v_n$.

Exercise 2.5. Assume $S : G \to GL(V)$ is a representation of a Lie group G, and its tangent representation dS is irreducible. Can we claim that S is also irreducible?

3. Covering of $SO_3(\mathbb{R})$ by SU_2

Let us introduce another widely used example. Denote by $Ad : SU_2 \to \mathrm{GL}(\mathfrak{su}_2)$ the representation defined via $Ad(A)(X) = AXA^{-1}$.

Question 3.1. Is Ad a representation of the Lie group SU_2?

Recall that the Lie group SU_2 can be defined as a set of the following matrices

$$SU_2 = \{\begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & a_1 - ia_2 \end{pmatrix} \mid a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1,\ a_1, a_2, a_3, a_4 \in \mathbb{R}\}? \tag{3.1}$$

An arbitrary element of the Lie algebra $\mathfrak{su}_2$ can be represented as

$$\begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}, x_1, x_2, x_3 \in \mathbb{R} \tag{3.2}$$

We get that $\det(X) = x_1^2 + x_2^2 + x_3^2$ is preserved by the map Ad. Thus we get that $\mathrm{im}(Ad) \subset O(\mathfrak{su}_2) \simeq O_3(\mathbb{R})$.

Exercise 3.2. Show that $\mathrm{im}(Ad) = SO(\mathfrak{su}_2) \simeq SO_3(\mathbb{R})$ and find the kernel.

That statement allows us to say that we have constructed a covering of $SO_3(\mathbb{R})$ by SU_2.

Exercise 3.3. Describe the map $d(Ad)$ in some basis.

4. Tensor products of Lie groups representations

Definition 4.1. Assume that G is a matrix Lie group. $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations of the matrix Lie group G. One can verify that the map $T \otimes S : G \to \mathrm{GL}(V \otimes U)$ is a continuous map. Therefore, it is a representation of the matrix Lie group G.

Exercise 4.2. Assume $T : G \to \mathrm{GL}(V), S : G \to \mathrm{GL}(U)$ are representations of a matrix Lie group G. Express the tangent representation $d(T \otimes S) : G \to \mathfrak{gl}(V \otimes U)$ using the tangent representations dT and dS.

The formula that we have obtained for the tangent representation $d(T \otimes S)$ is taken as a definition of the tensor product of two representations of a Lie algebra, even when those representations are not tangent representations of representations of a Lie group. One uses the notation

$$d(T \otimes S) = dT \otimes dS \tag{4.1}$$

This may lead to confusion because the tensor product in the RHS is understood by the formula we have obtained.

Exercise 4.3. Let V_n be the complex vector space of homogeneous polynomials of degree n in variables x and y. Recall the definition of the representation $T_n : SL_2(\mathbb{C}) \to GL(V_n)$ by

$$\left(\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) P\right)(x,y) := P(ax+cy, bx+dy). \tag{4.2}$$

Is the representation $S^n(T_1)$ isomorphic to the representation T_n ?

Exercise 4.4. (A special case of Clebsch–Gordan coefficients) Denote by $\tau_n : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_n)$ the representation of the Lie algebra $sl_2(\mathbb{C})$ corresponding to the highest weight module with the highest weight n. Find the decomposition of the representation $\tau_n \otimes \tau_m$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into the sum of its irreducible representations.

That decomposition will imply that the representation $\tau_n \otimes \tau_m$ is completely reducible. In the next lecture, we will show that every representation of the Lie group $SL_2(\mathbb{C})$ and its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is completely reducible. Here we are going to take that for granted.

That question is obviously equivalent to finding the following decomposition.

Exercise 4.5. Let V_n be the complex vector space of homogeneous polynomials of degree n in variables x and y. Define the map $T_n : SL_2(\mathbb{C}) \to GL(V_n)$ by

$$\left(\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) P\right)(x,y) := P(ax+cy, bx+dy). \tag{4.3}$$

Find the decomposition of the representation $T_n \otimes T_m$ of the matrix Lie group $SL_2(\mathbb{C})$ into the sum of its irreducible representations. Hint: We can compute the trace of the representation T_n, $T_n \otimes T_m$ at the element

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \tag{4.4}$$

and find the conditions on the multiplicities.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 8.2, 8.3, 11.1;
- Hall, Sections 3.1, 4.6;
- Vinberg, Section 10, 11.