LECTURE 13. IRREDUCIBLE REPRESENTATIONS OF THE LIE ALGEBRA OF THE LIE GROUP $SL_3(\mathbb{C})$

Please consider every proposition in this text to be an exercise.

1. Weights and roots

Let us recall the structure of an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module.
Put

$$E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \tag{1.1}$$

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{1.2}$$

$$F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

Proposition 1.1. *For every non-negative integer n there exists a unique irreducible $(n+1)$-dimensional representation ρ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In the space V of that representation we can pick a basis $v_0, v_1, \ldots, v_n$ with the following properties:*

$$\rho(H)v_k = (n-2k)v_k, \quad \forall 0 \le k \le n; \tag{1.4}$$

$$\rho(E)v_k = k(n+1-k)v_{k-1}, \quad \forall 1 \le k \le n, \tag{1.5}$$

$$\rho(E)v_0 = 0; \tag{1.6}$$

$$\rho(F)v_k = v_{k+1}, \quad \forall 0 \le k \le n-1, \tag{1.7}$$

$$\rho(F)v_n = 0. \tag{1.8}$$

The number n is called the highest weight of the representation ρ, the vector v_0 is called the highest weight vector, the representation ρ (or, the vector space V with the action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$) is called the highest weight $\mathfrak{sl}_2(\mathbb{C})$-module.

Now we are going to study the irreducible $\mathfrak{sl}_3(\mathbb{C})$-module by making a similar construction.

Definition 1.2. Let us denote

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{1.9}$$

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.10}$$

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.11}$$

Exercise 1.3. Find the eigenvectors and the eigenvalues of the elements H_1 and H_2 in the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.

Date: December 8, 2024.

Definition 1.4. Assume $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ is a representation. Then a pair $\beta = (b_1, b_2)$ is called a weight (of a vector v in the representation π) if $\exists v \in V, v \neq 0$ such that

$$\pi(H_1)(v) = b_1 v, \quad \pi(H_2)(v) = b_2 v. \tag{1.12}$$

Denote by $\alpha_1 = (2, -1)$, $\alpha_2 = (-1, 2)$ and $\alpha_3 = (1, 1)$ the weights of the Lie algebra elements E_1, E_2 and E_3 respectively in the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$. They are called roots.

Note that the weights of the elements F_1, F_2 and F_3 in the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$ are equal to $-\alpha_1$, $-\alpha_2$ and $-\alpha_3$ respectively.

Definition 1.5. We say that a weight β is higher than a weight γ if there exist non-negative real numbers p and q such that

$$\beta - \gamma = p\alpha_1 + q\alpha_2. \tag{1.13}$$

Proposition 1.6. *If $v \in V$ is a vector with the weight μ then $\pi(E_i)v$ is also a weight vector with the weight $\mu + \alpha_i$ and $\pi(F_i)v$ is a weight vector with the weight $\mu - \alpha_i$.*

Proposition 1.7. *Assume $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ is a representation. Then there exists a weight vector in V.*

In order to show this proposition, we may use the diagonaizability of $\pi(H_1)$ first. After that we can consider the action of $\pi(H_2)$ on a suitable subspace of highest weight vector wrt $\pi(H_1)$.

Actually we can prove even stronger result.

Proposition 1.8. *Assume $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ is an irreducible representation. Then V can be represented as a direct sum of its weight subspaces.*

Hint: We may consider the direct sum of all weight subspaces and show that it is an invariant subspace of the representation π.

2. Classification Theorem

We need to prove a useful lemma first.

Proposition 2.1. *Assume that $\mathfrak{g}$ is a Lie algebra, $x_1, \ldots, x_k$ is a set of generators of $\mathfrak{g}$, $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation. Then for any sequence $i_1, i_2, \ldots, i_m$ of integers from 1 to k we can represent the composition of the endomorhisms as*

$$\pi(x_{i_1}) \ldots \pi(x_{i_m}) = \sum_{j_1 + j_2 + \cdots + j_k \leq m} a_{j_1, \ldots, j_m} (\pi(x_1))^{j_1} \ldots (\pi(x_k))^{j_k} \tag{2.1}$$

Proposition 2.2. *Assume $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ is an irreducible representation, then it has a unique highest weight $\mu = (m_1, m_2)$. Those numbers m_1, m_2 are non-negative integers. The subspace of vectors with the weight μ is one-dimensional.*

Moreover every irreducible representation is uniquely determined by its highest weight.

Proposition 2.3. *Two irreducible representations with the same highest weight are isomorphic.*

The following claim might be useful.

Proposition 2.4. *Assume that $v_1, \dots, v_k$ are weight vectors with pairwise distinct weights $\beta_1, \dots, \beta_k$ respectively and*

$$v = \sum_{i=1}^{k} v_i \tag{2.2}$$

is also a weight vector with the weight μ. Then $\exists j$ such that $\mu = \beta_j$ and $v_i = 0$, $\forall 1 \leq i \leq k, i \neq j$.

Now we are going to show that for every pair of non-negative integers (m_1, m_2) there exists a corresponding irreducible representation.

Exercise 2.5. Denote by $St : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^3)$ the standard representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Show that it is irreducible and find its highest weight.

Exercise 2.6. Show that the representation $\Lambda^2(St)$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is irreducible and find its highest weight.

Exercise 2.7. Find the highest weight of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.

Proposition 2.8. *For every pair (m_1, m_2) of non-negative integers there exists an irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight (m_1, m_2)*

3. Connected Lie groups

Proposition 3.1. *Assume that G and H are matrix Lie groups, G is connected. Then every homomorphism $F : G \to H$ is uniquely determined by $dF : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$.*

Proof. The proof uses the result that a solution of a linear system of differential equations is uniquely determined by initial conditions. For an arbitrary $g \in G$ denote by $A : [0, 1] \to G$ a path connecting Id and g, $A(0) = \mathrm{Id}, A(1) = g$. Then $(A(t))^{-1}A'(t) \in T_{\mathrm{Id}}G$, $\forall t \in [0, 1]$. Denote the expression $(A(t))^{-1}A'(t)$ by $X(t)$.

Exercise 3.2. Show that then

$$dF\Big|_{A(t)}(A'(t)) = F(A(t))dF\Big|_{\mathrm{Id}}(X(t)) \tag{3.1}$$

Put

$$B(t) := F(A(t)), Y(t) := dF\Big|_{\mathrm{Id}}(X(t)). \tag{3.2}$$

Exercise 3.3. Show that

$$B'(t) = B(t)Y(t) \tag{3.3}$$

The solution of that system linear differential equations is uniquely determined by the initial condition $B(0) = \mathrm{Id}$.

□

We have previously shown that if the tangent representation dT is irreducible then the representation T has to be irreducible as well. Now let us study the reverse statement.

Proposition 3.4. *Assume that G is a connected matrix Lie group, $T : G \to GL(V)$ is its representation. If T is irreducible then the representation dT is also irreducible.*

Hint: One may use an approach similar to the proof of Proposition 1.1.

Proposition 3.5. *Assume that G and H are matrix Lie groups, G is simply connected (i.e. $\pi_1(G) = 0$). Then for every homomorphism $\phi : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$ of the Lie algebras there exists a homomorphism $F : G \to H$ of the Lie groups such that $dF = \phi$.*

We are going to take that result for granted.

Chapters from the textbooks relevant for the lecture:

- Fulton, Harris, Sections 8.2, 8.3, 11.1, 11.2, 12, 13;
- Hall, Sections 6.1 - 6.4.
- Vinberg, Section 10, 11.