LECTURE 14. COMPACT GROUPS, PETER-WEYL THEOREM. COMPLETE REDUCIBILITY.

1. Compact Lie groups

Proposition 1.1. *Assume that G is a compact matrix Lie groups, $T : G \to \mathrm{GL}(V)$ is its complex representation. Then there exist an inner product on V such that $T(g)$ is a unitary operator for every $g \in G$. If T is irreducible then the inner product is uniquely determined up to multiplication by a positive scalar.*

We are going to take that result for granted.

Proposition 1.2. *Assume that G is a compact matrix Lie groups, $T : G \to \mathrm{GL}(V)$ is its complex representation. Then T is completely reducible.*

2. Connected Lie groups

Proposition 2.1. *Assume that G and H are matrix Lie groups, G is connected. Then every homomorphism $F : G \to H$ is uniquely determined by $dF : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$.*

Proof. The proof uses the result that a solution of a linear system of differential equations is uniquely determined by initial conditions. For an arbitrary $g \in G$ denote by $A : [0, 1] \to G$ a path connecting Id and g, $A(0) = \mathrm{Id}, A(1) = g$. Then $(A(t))^{-1}A'(t) \in T_{\mathrm{Id}}G$, $\forall t \in [0, 1]$. Denote the expression $(A(t))^{-1}A'(t)$ by $X(t)$.

Exercise 2.2. Show that then

$$dF\Big|_{A(t)}(A'(t)) = F(A(t))dF\Big|_{\mathrm{Id}}(X(t)) \tag{2.1}$$

Put

$$B(t) := F(A(t)), Y(t) := dF\Big|_{\mathrm{Id}}(X(t)). \tag{2.2}$$

Exercise 2.3. Show that

$$B'(t) = B(t)Y(t) \tag{2.3}$$

The solution of that system linear differential equations is uniquely determined by the initial condition $B(0) = \mathrm{Id}$.

□

We have previously shown that if the tangent representation dT is irreducible then the representation T has to be irreducible as well. Now let us study the reverse statement.

Proposition 2.4. *Assume that G is a connected matrix Lie group, $T : G \to GL(V)$ is its representation. If T is irreducible then the representation dT is also irreducible.*

Hint: One may use an approach similar to the proof of Proposition 2.1.

Date: December 8, 2024.

Proposition 2.5. *Assume that G and H are matrix Lie groups, G is simply connected (i.e. $\pi_1(G) = 0$). Then for every homomorphism $\phi : T_{\mathrm{Id}}G \to T_{\mathrm{Id}}H$ of the Lie algebras there exists a homomorphism $F : G \to H$ of the Lie groups such that $dF = \phi$.*

We are going to take that result for granted.

3. Complete reducibility of the representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

Proposition 3.1. *The operation of restriction provides a natural bijection between complex representations of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and real representations of the Lie algebra $\mathfrak{su}_n$ in complex vector spaces.*

To prove this proposition.do that we may establish that the comlexification of the Lie algebra $\mathfrak{su}_n$ is naturally isomorphic to $\mathfrak{sl}_n(\mathbb{C})$.

Proposition 3.2. *Every representation of the Lie algebra $\mathfrak{su}_2$ is completely reducible.*

Here we may use that the group SU_2 is connected and simply connected.

Proposition 3.3. *Every complex representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is completely reducible.*

Chapters from the textbooks relevant for the lecture:

- Fulton, Harris, Sections 8.2, 8.3, 11.1;
- Hall, Sections 3.5, 4.6, 4.7;
- Vinberg, Section 10, 11.