PROBLEM SET 1, DUE SEPTEMBER 21

1. Definition of a representation

Problem 1.1. *Find all 1-dimensional representations of the cyclic group $\mathbb{Z}_n$*
a) over $\mathbb{R}$, b) over $\mathbb{C}$.

Problem 1.2. *When the map $T : \mathbb{Z} \to \mathrm{Mat}(2 \times 2, \mathbb{R})$ given by the formula*

a) $T(n) = \begin{pmatrix} an & 0 \\ 0 & bn \end{pmatrix}$ *b)* $T(n) = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ *c)* $T(n) = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix}$

d) $T(n) = \begin{pmatrix} 1 & a^n \\ 0 & 1 \end{pmatrix}$ *e)* $T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}$, a, b, $\alpha \in \mathbb{R}$.

is a representation of the group $\mathbb{Z}$ over the field $\mathbb{R}$?

Problem 1.3. *Describe the non-trivial invariant subspaces for the representations from Problem 1.2. Which of those representations are irreducible or completely reducible?*

Problem 1.4. *Consider the representation $T : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}^2)$ given by the formula*

$$T(k) = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin\frac{2\pi k}{n}) \\ -\sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Find all its non-trivial invariant subspaces.

Problem 1.5. *Which of the representations from Problem 1.2 are pairwise isomorphic?*

Problem 1.6. *Does there exist an irreducible real representation T of the group $\mathbb{Z}_2$ such that $\dim T \geq 2$?*

Problem 1.7. *Does there exist a real representation T of the group $\mathbb{Z}_3 = \{0, 1, 2\}$ such that*

$$T(1) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}?$$

Problem 1.8. *Does there exist a representation of a finite group that is not completely reducible?*

Problem 1.9. *Define a representation T of the symmetric (permutation) group S_n in $\mathbb{C}^n$ by*

$$T(\sigma)((x_1, \ldots, x_n)) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

for $\sigma \in S_n$. Find all non-trivial invariant subspaces of the representation T.

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