PROBLEM SET 2, DUE SEPTEMBER 28

2. Schur's Lemma.

Problem 2.1. *Describe the algebras of endomorphisms of representations from Problem 1.2 of Problem Set 1.*

Problem 2.2. *Find all one-dimensional complex representations of the group* $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Problem 2.3. *The dihedral group* D_n *is generated by a rotation* a *and a reflection* s *such that* $a^n = s^2 = e$, $sas = a^{-1}$. *Find all 1-dimensional complex representations of the group* D_n. *Note that the answer depends on the parity of* n.

Problem 2.4. *Show that* $[S_n, S_n] = A_n$, *where* $A_n := sgn^{-1}(1)$.

Problem 2.5. *Find all 1-dimensional complex representations of the group* A_4.

Problem 2.6. *Consider an equilateral triangle on the plane with the center at the point* $(0, 0)$. *The symmetric group* S_3 *acts on this triangle by permutations of its vertices. Show that the corresponding 2-dimensional representation* T *of the group* S_3 *over* $\mathbb{R}$ *is irreducible. Now let us consider those real-valued matrices* $T(\sigma)$, $\sigma \in S_3$ *as complex-valued. Show that the corresponding complex representation* $T^{\mathbb{C}}$ *(complexification) of the group* S_3 *over* $\mathbb{C}$ *is also irreducible.*

Problem 2.7. *Consider a regular tetrahedron in the Euclidean space* $\mathbb{R}^3$ *with the center at the origin. Denote by* $O(T)$ *the set of isometries of* $\mathbb{R}^3$ *preserving the tetrahedron. Also put* $SO(T) = \{A \in O(T) \mid \det(A) > 0\}$. *Show that*

$$O(T) \simeq S_4, \quad SO(T) \simeq A_4. \tag{2.1}$$

Problem 2.8. *Consider a cube in the Euclidean space* $\mathbb{R}^3$ *with the center at the origin. Denote by* $O(C)$ *the set of isometries of* $\mathbb{R}^3$ *preserving the cube. Also put* $SO(C) = \{A \in O(C) \mid \det(A) > 0\}$. *Show that*

$$O(C) \simeq S_4 \times \mathbb{Z}_2, \quad SO(C) \simeq S_4. \tag{2.2}$$

Problem 2.9. *Show that the three-dimensional real representations of the group* A_4 *determined by the motions of the regular tetrahedron and the cube are irreducible.*

Problem 2.10. *Is the representation of the group* S_4 *in* $\mathbb{R}^3$ *determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?*

Problem 2.11. *Is the representation of the group* A_4 *in* $\mathbb{R}^3$ *determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?*

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