PROBLEM SET 3, DUE OCTOBER 5

3. Maschke's Theorem. Tensor product of representations.

Problem 3.1. *Assume that* $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{C}^2)$ *and* $S : \mathbb{Z} \to \mathrm{GL}(\mathbb{C}^2)$ *are defined as*

$$T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}, \quad S(n) = \begin{pmatrix} \exp(in\alpha) & 0 \\ 0) & \exp(-in\alpha) \end{pmatrix}.$$

Describe $\mathrm{Mor}(T, S)$.

Problem 3.2. *Assume that* $T : \mathbb{Z}_4 \to \mathrm{GL}(\mathbb{C}^3)$ *and* $S : \mathbb{Z}_4 \to \mathrm{GL}(\mathbb{C}^3)$ *are defined as*

$$T(n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & i^n \end{pmatrix}, \quad S(n) = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & i^n & 0 \\ 0 & 0 & i^n \end{pmatrix}.$$

Describe $\mathrm{Mor}(T, S)$ *and* $\mathrm{End}(S)$.

Problem 3.3. *Is it true that every irreducible complex representation of an abelian group is one-dimensional?*

Problem 3.4. *Suppose that the group* G *is finite and* $T : G \to \mathrm{GL}(\mathbb{C}^2)$ *is a representation. Suppose* $\exists g, h \in G$ *such that* $T(g)T(h) \neq T(h)T(g)$. *Can the representation* T *be not irreducible?*

Problem 3.5. *Let* $k \in \mathbb{Z}$. *Define a 2-dimensional complex representation* T_k *of the group* D_n *as*

$$T_k(a) = \begin{pmatrix} \cos\frac{2\pi k}{n} & \sin\frac{2\pi k}{n} \\ -\sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{pmatrix} \qquad T_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Check that the representation T_k *is well-defined for all integer* k. *When is the representation* T_k *irreducible?*

Problem 3.6. *Define* $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^3)$ *as*

$$T(n) = \begin{pmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that T *is a representation. Find the matrix of* $T^*(n)$ *in the dual basis.*

Problem 3.7. *Let* $T : G \to \mathrm{GL}(V)$ *be a representation. Is it always true that the representations* T *and* $(T^*)^*$ *are isomorphic?*

Problem 3.8. *Let* T *be an irreducible representation of a group* G. *Is it possible that the representation* T^* *is not irreducible?*

Problem 3.9. *Let* $T_1, \ldots, T_n$ *be the set of all pairwise non-isomorphic one-dimensional complex representations of* G. *Show that this set forms a group, where the tensor product stands for the multiplication, and the inverse element is the dual representation. Describe that group for* $G = \mathbb{Z}_n$, S_n.

Problem 3.10. *Let* T *be a representation of a finite group* G. *Is it true that the representation* $T \otimes T^*$ *of the group* G *always has a one-dimensional invariant subspace ?*

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Problem 3.11. *Let* $\mathrm{char}\mathbb{F} \neq 2$. *Is it always true that*

$$T \otimes T = S^2T \oplus \Lambda^2 T?$$