# PROBLEM SET 5, DUE OCTOBER 19

## 5. Group algebra.

**Problem 5.1.** *Find the multiplicities of the decomposition of every tensor product of two irreducible complex representations (they may be equal) of the group $A_4$ into the sum of irreducible ones.*

**Problem 5.2.** *Find the multiplicities of the decomposition of every tensor product of two irreducible complex representations (they may be equal) of the group $S_4$ into the sum of irreducible ones.*

**Problem 5.3.** *For every irreducible complex representation $T$ of the group $S_4$ find the multiplicities of the decomposition of the representations $S^2T$ and $\Lambda^2T$ into the sum of irreducible ones.*

**Problem 5.4.** *Find the multiplicities of the decomposition of every tensor product of two irreducible complex representations (they may be equal) of the group $D_n$ into the sum of irreducible ones.*

**Problem 5.5.** *For every irreducible complex representation $T$ of the group $D_n$ find the multiplicities of the decomposition of the representations $S^2T$ and $\Lambda^2T$ into the sum of irreducible ones.*

**Problem 5.6.** *Consider a set $F$ of complex functions on the set $M$ of the faces of the cube. It is clear that $F$ is a vector space with respect to the addition and the multiplication by complex numbers defined by the following formulas:*

$$(\lambda \cdot f)(x) := \lambda(f(x)), \qquad (f+g)(x) := f(x) + g(x), \qquad \forall f, g \in F, \forall x \in M, \forall \lambda \in \mathbb{C}.$$

*Recall that $S_4 \cong SO(C)$, which is the group of orientation-preserving isometries of the cube. For arbitrary elements $g \in S_4$, $x \in M$ denote by $gx$ the image of the face $x$ after applying the rotation from $SO(C)$ corresponding to $g$. Denote by $L$ a representation of the group $S_4$ in the vector space $F$ given by the following formula*

$$(L(g)f)(x) := f(g^{-1}x), \qquad \forall f \in F, \forall x \in M, \forall g \in S_4.$$

*Find the multiplicities of the decomposition of the representation $L$ into the sum of irreducible ones.*

**Problem 5.7.** *Describe minimal invariant subspaces of the representation $L$ from the previous problem.*

**Problem 5.8.** *Consider a set $F$ of complex functions on the set $M$ of the edges of the cube. It is clear that $F$ is a vector space with respect to the addition and the multiplication by complex numbers defined by the following formulas:*

$$(\lambda \cdot f)(x) := \lambda(f(x)), \qquad (f+g)(x) := f(x) + g(x), \qquad \forall f, g \in F, \forall x \in M, \forall \lambda \in \mathbb{C}.$$

*Recall that $S_4 \cong SO(C)$, which is the group of orientation-preserving isometries of the cube. For arbitrary elements $g \in S_4$, $x \in M$ denote by $gx$ the image of the edge $x$ after applying the rotation from $SO(C)$ corresponding to $g$. Denote by $L$ a representation of the group $S_4$ in the vector space $F$ given by the following formula*

$$(L(g)f)(x) := f(g^{-1}x), \qquad \forall f \in F, \forall x \in M, \forall g \in S_4.$$

*Date*: October 16, 2024.

*Find the multiplicities of the decomposition of the representation $L$ into the sum of irreducible ones.*

**Problem 5.9.** *Describe minimal invariant subspaces of the representation $L$ from the previous problem.*

**Problem 5.10.** *Are the associative algebras $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_4]$ isomorphic?*