PROBLEM SET 6, DUE OCTOBER 26

6. Algebraic isomorphism of a complex group algebra.

Problem 6.1. *Let T be an irreducible real representation of an abelian group G. Is it always true that $|\dim T| \leq 2$?*

Problem 6.2. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(V)$ are complex representations. Show that the representation $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ is irreducible if and only if both T and S are irreducible.*

Problem 6.3. *Let n be even. Find the multiplicities of the decomposition of every tensor product of two irreducible complex representations (they may be equal) of the group D_n into the sum of irreducible ones.*

Problem 6.4. *Consider the group*

$$G = \langle a, b \,|\, a^5 = b^4 = e, bab^{-1} = a^2 \rangle.$$

Find its complex character table.

Problem 6.5. *Does there exist a group G such that $|G| = 24$ and $[G, G] = G$?*

Problem 6.6. *Let G be a finite group. Consider the left regular representation*

$$LReg : G \to \mathrm{GL}(\mathbb{C}[G]). \tag{6.1}$$

Describe the minimal invariant subspace of $\mathbb{C}[G]$ corresponding to the trivial subrepresentation.

Problem 6.7. *A representation T is called faithful if $\ker(T) = \{e\}$. How do we determine whether a complex representation of a finite group is faithful using the values of its character?*

Problem 6.8. *Suppose that T is a faithful complex representation of a finite group G. Then for every irreducible complex representation $S \in \widehat{G}$ there exists a non-negative integer k such that S enters in the decomposition of $T^{\otimes k}$ with a non-zero multiplicity.*

Problem 6.9. *Let g be an element of order 2 in the finite group G and let χ be the character of a complex representation of G. Show that $\chi(g)$ is an integer and exactly one of two conditions hold: either*

$$\chi(g) \equiv \chi(e) (mod 4), \tag{6.2}$$

or G has a normal subgroup of index 2.

Problem 6.10. *Show that for every normal subgroup H of a finite group G there exist irreducible complex characters $\chi_1, \ldots, \chi_k$ such that*

$$H = \bigcap_{j=1}^{k} \{g \in G \mid \chi_j(g) = \chi_j(e)\}$$

Problem 6.11. *A group is called simple if it has no non-trivial normal subgroups. How can we check if a finite group is simple using its complex character table?*

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Problem 6.12. *Consider a set F of complex functions on the set M of the edges of the cube. It is clear that F is a vector space with respect to the addition and the multiplication by complex numbers defined by the following formulas:*

$$(\lambda \cdot f)(x) := \lambda(f(x)), \qquad (f+g)(x) := f(x) + g(x), \qquad \forall f, g \in F, \forall x \in M, \forall \lambda \in \mathbb{C}.$$

Recall that $S_4 \cong SO(C)$, which is the group of orientation-preserving isometries of the cube. For arbitrary elements $g \in S_4$, $x \in M$ denote by gx the image of the edge x after applying the rotation from $SO(C)$ corresponding to g. Denote by L a representation of the group S_4 in the vector space F given by the following formula

$$(L(g)f)(x) := f(g^{-1}x), \qquad \forall f \in F, \forall x \in M, \forall g \in S_4.$$

Find the multiplicities of the decomposition of the representation L into the sum of irreducible ones. Describe its minimal invariant subspaces.