PROBLEM SET 10, DUE NOVEMBER 23

10. Lie Algebras.

Problem 10.1. *Let G be a matrix Lie group. Show that its connected component of the identity is a normal subgroup of G.*

Problem 10.2. *Show that*

$$\mathrm{Sp}(1,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}), \quad \mathrm{Sp}(1,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C}).$$

Problem 10.3. *Is the matrix Lie group $\mathrm{GL}(n,\mathbb{C})$ connected?*

Problem 10.4. *Is the matrix Lie group $\mathrm{GL}(n,\mathbb{R})$ connected?*

Problem 10.5. *Is the matrix Lie group $SU(n)$ connected?*

Problem 10.6. *Describe the Lie algebra of the Lie group of invertible diagonal matrices.*

Problem 10.7. *Find the Lie algebra of the Lie group invertible matrices of the form* $\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)$, *where the diagonal blocks are squares.*

Problem 10.8. *Describe the Lie algebras and find their dimensions for the Lie groups $UT_n(\mathbb{F}), UUT_n(\mathbb{F})$.*

Problem 10.9. *Find the Lie algebras and their dimensions for the Lie groups $SO_n(\mathbb{R})$ and $SO_n(\mathbb{C})$.*

Problem 10.10. *Find the Lie algebra and its dimensions of the Lie groups SU_n.*

Problem 10.11. *Find the Lie algebra and its dimension of the Lie group $Sp_n(\mathbb{F})$.*

Problem 10.12. *Show that the map $\exp : \mathfrak{su}_2 \to SU_2$ is not injective.*

Problem 10.13. *Show that the map $\exp : \mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ is not surjective.*

Problem 10.14. *Consider the set D of differential operators of the form $\alpha(x)\frac{d}{dx}$, $\alpha(x) \in C^\infty(\mathbb{R})$, where $\alpha(x)\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is defined by*

$$((\alpha(x)\frac{d}{dx})(f))(x) = \alpha(x)f'(x)$$

The set D is a Lie algebra with respect to the operation of standard commutator. Show that

$$\mathrm{span}\langle \frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx} \rangle$$

is a Lie subalgebra of D. Prove that this subalgebra is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

Date: November 20, 2024.