PROBLEM SET 12, DUE DECEMBER 7

12. Lie Algebras representations.

Problem 12.1. *Let V_n be the complex vector space of homogeneous polynomials of degree n in variables x and y. Define the map $T_n : SL_2(\mathbb{C}) \to GL(V_n)$ by*

$$\left(\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) P\right)(x,y) := P(ax+cy, bx+dy). \tag{12.1}$$

Is the representation $S^n(T_1)$ isomorphic to the representation T_n ?

Problem 12.2. *Denote by $\tau_n : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_n)$ the representation of the Lie algebra $sl_2(\mathbb{C})$ corresponding to the highest weight module with the highest weight n. Find the decomposition of the representation $\tau_n \otimes \tau_m$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into the sum of its irreducible representations.*

Problem 12.3. *Consider the restriction R of the adjoint representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ to its Lie subalgebra $\mathfrak{sl}_2(\mathbb{C})$. Find the multiplicity of each irreducible $\mathfrak{sl}_2(\mathbb{C})$-module τ_n (with the highest weight n) in the decomposition of R into the sum of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.*

Problem 12.4. *For a Lie group G and a representation $T : G \to GL(V)$ put*

$$V^G = \{v \in V \mid T(g)v = v, \forall g \in G\}.$$

Find $\dim\left((V_1^{\otimes 2n})^{SL_2(\mathbb{C})}\right)$, where $T_1 : SL_2(\mathbb{C}) \to \mathrm{GL}(V_1)$ is the standard two-dimensional representation of the Lie group $SL_2(\mathbb{C})$.

Problem 12.5. *Decompose the representation $\Lambda^2(\tau_4)$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into the sum of its irreducible representations.*

Problem 12.6. *Decompose the representation $S^2(\tau_4)$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into the sum of its irreducible representations.*

Problem 12.7. *Denote by St the standard representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Is the representation $\Lambda^2 St$ irreducible?*

Problem 12.8. *Is the representation $\Lambda^2(ad_{\mathfrak{so}_3(\mathbb{R})})$ of the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ irreducible ?*

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