PROBLEM SET 13, DUE DECEMBER 14

13. Weights and roots.

Let ρ be a representation of the Lie algebra $sl_2(\mathbb{C})$. The endomorphism

$$K_\rho = \rho(E)\rho(F) + \rho(F)\rho(E) + \rho(H)^2/2$$

is called a Casimir element.

Problem 13.1. *Show that K_ρ commutes with the action of $sl_2(\mathbb{C})$.*

Problem 13.2. *Find K_ρ when ρ is the highest weight $\mathfrak{sl}_2(\mathbb{C})$-module τ_n.*

Problem 13.3. *Show that any two irreducible representations of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ with the same highest weight are isomorphic.*

Problem 13.4. *Show that the representation $\Lambda^2(St)$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is irreducible and find its highest weight.*

Problem 13.5. *Find the highest weight of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.*

Problem 13.6. *Construct more explicitly an irreducible representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight $(m, 0)$ for every integer $m \geq 0$,*

Problem 13.7. *Let π be an irreducible representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight (m_1, m_2). Find the highest weight of the representation π^* if it is irreducible.*

Problem 13.8. *For every pair (m_1, m_2) of non-negative integers there exists an irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight (m_1, m_2).*

Problem 13.9. *Find the dimension of an irreducible representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight $(0, m)$.*

Problem 13.10. *Find the dimension of an irreducible representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight $(2, 1)$.*

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