"Basic Representation Theory", Fall 2024.

Course description. Welcome to the Basic Representation Theory course! Representation Theory studies how a given group may act in vector spaces. It is a fundamental tool to study groups using linear algebra. But even more importantly the structure of representations allows us to construct basic mathematical models in Mathematical Physics and other applications. The course introduces basic concepts and results of the classical theory of complex representations of finite groups and simple examples of representations of Lie groups and Lie algebras.

Program of the Course.

1. Alongside the discussion of representations we are going to recall some (multi)linear algebra concepts and theorems: Jordan canonical form theorem, bilinear forms, inner product spaces, spectral theorem for unitary operators, tensor products of vector spaces, symmetric and exterior algebras.
2. Examples of representations. Subrepresentations. Invariant subspaces. Irreducible representations. Direct sum of representations.
3. Morphisms of representations. Schur's lemma.
4. Tensor product of representations.
5. Complete reducibility. Maschke's theorem. Uniqueness of the decomposition of completely reducible representation into a sum of irreducible ones.
6. Characters of representations, their properties. Conjugacy classes. Group algebra. Central functions. Regular representation.
7. Orthogonality of complex matrix elements. Group algebra of a finite group as a product of matrix algebras, corresponding to irreducible representations.
8. Number of irreducible complex characters. Orthogonality relations.
9. Examples of complex character tables: abelian groups, dihedral group Dn, groups S3, S4, A4. Decomposition of tensor products of irreducible representations.
10. * Representations of symmetric groups, Young diagrams.
11. Examples of Lie groups. Definition of a matrix Lie group. Connected Lie groups.
12. Covering of SO(3,R) by SU(2).
13. Lie algebra is the tangent space of a Lie group at the identity element. Definition of an abstract Lie algebra.
14. Representations of the Lie algebra sl(2,C).
15. The connection between representations of Lie groups and Lie algebras. Polynomial representation of the Lie group sl(2,C).
16. Tensor product of representations of Lie algebras. Clebsch-Gordan decomposition for the Lie algebra sl(2,C).

17. Compact groups and their representations. Peter-Weyl theorem.
18. Highest weight sl(3,C)-modules.
19. * Complex simple Lie algebras and their irreducible representations.

* - if time allows

Recommended Textbooks.

1. W. Fulton, J. Harris, Representation Theory. A First Course.

2. E.B. Vinberg, Linear Representations of Groups.

3. G.James, M.Liebeck, Representations and Characters of Groups. (for items 1-5 of the program)

4. B.C. Hall, Lie Groups, Lie Algebras, and Representations. (for items 6-10 of the program)

Grading Rules. You should write a solution to at least two problems before each seminar and two exercises before each lecture. You will get 1 point for every correct solution but no more than two points for each lecture or seminar. Additional bonus points may be awarded for the shortest and most elegant solutions or solutions for the most complicated exercises. Both the midterm and the final exam are oral exams, open books, and open notes. The midterm contains 5 problems for 3 points each, and the final – 5 problems for 4 points each. Also, you get one bonus point every time you are the first to correct the professor's mistake/mistype during the lecture. The grading table has the following form:

A+ 90%

A 85%

A- 80%

B+ 75%

B 70%

B- 65%

C+ 55%

C 50%

C- 45%

D 35%

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