LECTURE 1. THE NOTION OF A REPRESENTATION.

Please treat each proposition in this text as an exercise.

1. Preliminary Definitions

To define a representation, we need to introduce our actors first. We start with the notion of a group.

Definition 1.1. A set G with the maps $* : G \times G \to G$ (notation $*(a, b) = ab$) and $inv : G \to G$ (notation $inv(a) = a^{-1}$) is called a group if the following properties hold:

(1) $(ab)c = a(bc), \quad \forall a, b, c \in G$;
(2) $\exists e \in G$ such that $ae = ea = a, \quad \forall a \in G$;
(3) $\forall a \in G, \exists b \in G$ such that $ab = ba = e$.

Question 1.2. Does the definition of a group imply that an identity element is unique?

Definition 1.3. Suppose that G and H are groups. A map $f : G \to H$ is called a group homomorphism if $f(g_1 g_2) = f(g_1) f(g_2)$, $\forall g_1, g_2 \in G$.

Question 1.4. Let $f : G \to H$ be a homomorphism. Is it always true that $f(e_G) = e_H$ and $f(g^{-1}) = (f(g))^{-1}$, $\forall g \in G$?

Definition 1.5. A group G is called abelian if $ab = ba, \quad \forall a, b \in G$.

Question 1.6. Let V be a vector space over a field $\mathbb{F}$. Show that the set of invertible linear maps from V to itself is a group with respect to the operation of maps composition.

Definition 1.7. Let V be a vector space over a field $\mathbb{F}$. The set of invertible linear maps from V to itself is called a general linear group and is denoted by $GL(V)$.

Question 1.8. Is $GL(V)$ a vector space with respect to the naturally defined addition and multiplication by scalars?

2. Definition of a Representation

Definition 2.1. Suppose that G is a group, $\mathbb{F}$ is a field, and V is a vector space over $\mathbb{F}$. Then a homomorphism

$$T : G \to GL(V) \tag{2.1}$$

is called a representation of the group G (in the vector space V) (over the field $\mathbb{F}$). The dimension $\dim T$ of the representation T is defined as the dimension of the vector space V.

Question 2.2. For any given group G and vector space V does there always exist a representation of G in V?

Exercise 2.3. Find all 1-dimensional representations of the cyclic group $\mathbb{Z}_n$:
a) over $\mathbb{R}$, b) over $\mathbb{C}$.

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Exercise 2.4. When is the map $T : \mathbb{Z} \to \mathrm{Mat}(2 \times 2, \mathbb{R})$ given by the formula

a) $T(n) = \begin{pmatrix} an & 0 \\ 0 & bn \end{pmatrix}$ b) $T(n) = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ c) $T(n) = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix}$

d) $T(n) = \begin{pmatrix} 1 & a^n \\ 0 & 1 \end{pmatrix}$ e) $T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}$ $a,\ b,\ \alpha \in \mathbb{R}$.

is a representation of the group $\mathbb{Z}$ over the field $\mathbb{R}$?

3. Sum of Two Representations. Invariant subspaces

Now let us define a sum of two representations. First, we recall a construction from linear algebra.

Definition 3.1. Suppose that V and U are vector spaces over $\mathbb{F}$. The Cartesian product $V \times U$ with respect to operations of component-wise addition and multiplication by scalars is called the (external) direct sum of V and U and is denoted by $V \oplus U$.

Now we can define the sum of two representations of the same group over the same field.

Definition 3.2. Assume that $T : G \to GL(V)$ and $S : G \to GL(U)$ are representations over $\mathbb{F}$. Then define the sum of representations T and S $T \oplus S : G \to GL(V \oplus U)$ via

$$(T \oplus S)(g)(v, u) := (T(g)v, S(g)u), \quad \forall g \in G, v \in V, u \in U. \tag{3.1}$$

We may suspect that a representation is a sum of representations of smaller dimensions. How should we proceed to find those smaller blocks? It would be natural to give the following definition.

Definition 3.3. Let $T : G \to GL(V)$ be a representation. A subspace $U \subset V$ is called invariant with respect to T if

$$T(g)u \in U, \quad \forall g \in G, u \in U. \tag{3.2}$$

If $U \neq \{0\}$, $U \neq V$ then U is called non-trivial.

Note that we can construct the representation $S : G \to GL(U)$, setting $S(g)$ to be the restriction of the automorphism $T(g)$ to the subspace U. S is called a subrepresentation of the representation T.

Exercise 3.4. Describe the non-trivial invariant subspaces for the representations from Exercise 2.4.

Exercise 3.5. Consider the representation $T : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}^2)$ given by the formula

$$T(k) = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\ -\sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Find all its non-trivial invariant subspaces.

Question 3.6. Assume $T : G \to GL(V)$ is a representation, $U \subset V$ is an invariant subspace, $u_1, \ldots, u_k$ is a basis of U, $u_1, \ldots, u_k, w_1, \ldots, w_m$ is a basis of V. We can express $T(g)$ in a block-diagonal form:

$$T(g) = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}. \tag{3.3}$$

How can we reformulate the fact that U is an invariant subspace in terms of $A(g), B(g), C(g), D(g)$?

4. Irreducible and Completely Reducible Representations

Definition 4.1. Assume $T : G \to GL(V)$ is a representation. T is called irreducible if it has no non-trivial invariant subspaces.

Definition 4.2. Assume $T : G \to GL(V)$ is a representation. T is called completely reducible if for any invariant subspace U there exist another invariant subspace W such that $V = U \oplus W$. Here the notation $\oplus$ stands for another (internal) direct sum.

Question 4.3. Is it true that any irreducible representation is completely reducible?

Question 4.4. Does there exist a representation that is not completely reducible?

Exercise 4.5. Define a representation T of the symmetric (permutation) group S_n in $\mathbb{C}^n$ by

$$T(\sigma)((x_1, \dots, x_n)) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

for $\sigma \in S_n$. Find all non-trivial invariant subspaces of the representation T.

5. Isomorphic Representations

We are going to discuss when is natural to call two representations $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ of the same group G over the same field $\mathbb{F}$ isomorphic.

Question 5.1. Let $T : G \to \mathrm{GL}(V)$ be a representation. Then V can also be called G-module, meaning that besides addition and scalar multiplication, we define an action of the group on G, i.e. a map $G \times V \to V$. Give a definition of a G-module, so that every G-module corresponds to some representation and vice versa.

Question 5.2. Let G be a group. Give a definition of isomorphic G-modules.

Question 5.3. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations over the same field $\mathbb{F}$. When would it be natural to say that T and S are isomorphic?

6. Morphisms of representations and Schur's Lemma

Definition 6.1. Assume that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. $A \in \mathrm{Hom}(V, U)$ is called a morphism of representations T and S if

$$AT(g) = S(g)A, \quad \forall g \in G. \tag{6.1}$$

We are going to denote the set of morphisms of representations T and S by $\mathrm{Mor}(T, S)$ (by $\mathrm{End}(T)$ if $S = T$).

Question 6.2. How do we reformulate the definition of a morphism of representations if we consider those representations as G-modules?

Question 6.3. Is $\mathrm{End}(T)$ a subspace of $\mathrm{End}(V)$?

Exercise 6.4. Describe all morphisms between two one-dimensional complex representations of the group $\mathbb{Z}_n$.

Definition 6.5. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. They are called isomorphic if $\exists A \in \mathrm{Iso}(V, U)$ such that $A \in \mathrm{Mor}(T, S)$ (A is called an isomorphism of representations in that case). The set of isomorphisms of representations T and S is denoted by $\mathrm{Iso}(T, S)$. The set of automorphisms of the representation T is denoted by $\mathrm{Aut}(T)$.

Lemma 6.6. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations, $A \in \mathrm{Mor}(T, S)$. Then*

a) $\ker A = \{v \in V \mid Av = 0\} \subset V$ *is an invariant subspace with respect to T;*

b) $\mathrm{im} A = \{Av \mid v \in V\} \subset U$ *is an invariant subspace with respect to S.*

Proposition 6.7. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are irreducible representations, $A \in \mathrm{Mor}(T, S)$. Then either $A = 0$ or $A \in \mathrm{Iso}(T, S)$.*

Proposition 6.8. *(Schur's Lemma) Suppose that $T : G \to \mathrm{GL}(V)$ is an irreducible complex representation, and $A \in \mathrm{End}(T)$. Then $\exists \lambda \in \mathbb{C}$ such that $A = \lambda \mathrm{Id}$.*

Question 6.9. Is the claim of Schur's Lemma still true over $\mathbb{R}$?

Definition 6.10. A vector space V over a field $\mathbb{F}$ with an operation $\cdot : V \times V \to V$ is called an algebra if the following properties hold:

(1) $a(b + c) = ab + ac, \quad \forall a, b, c \in V$;
(2) $(b + c)a = ba + bc, \quad \forall a, b, c \in V$;
(3) $\lambda(ab) = (\lambda a)b = a(\lambda b), \quad \forall a, b \in V, \forall \lambda \in \mathbb{F}$.

An algebra V is called associative if $a(bc) = (ab)c$, $\forall a, b, c \in V$. An associative algebra V is called a division algebra if $(V \setminus \{0\}, \cdot)$ is a group.

Question 6.11. Assume $T : G \to \mathrm{GL}(V)$ is an irreducible representation over $\mathbb{F}$. Does $\mathrm{End}(T)$ have to be a division algebra over $\mathbb{F}$?

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Section 1.1, 1.2;
- James, Liebeck, Chapters 1 - 5, 7 - 9;
- Vinberg, Sections 1.1 - 1.4.