LECTURE 2. SCHUR'S LEMMA. MASCHKE'S THEOREM.

Please treat each proposition in this text as an exercise.

1. Morphisms of representations and Schur's Lemma

Definition 1.1. Assume that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. $A \in \mathrm{Hom}(V, U)$ is called a morphism of representations T and S if

$$AT(g) = S(g)A, \quad \forall g \in G. \tag{1.1}$$

We will denote the set of morphisms of representations T and S by $\mathrm{Mor}(T, S)$ (by $\mathrm{End}(T)$ if $S = T$).

Question 1.2. How can we reformulate the definition of a morphism of representations if we consider those representations as G-modules?

Question 1.3. Is $\mathrm{End}(T)$ a subspace of $\mathrm{End}(V)$?

Exercise 1.4. Describe all morphisms between two one-dimensional complex representations of the group $\mathbb{Z}_n$.

Definition 1.5. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. They are called isomorphic if $\exists A \in \mathrm{Iso}(V, U)$ such that $A \in \mathrm{Mor}(T, S)$. In this case,A is called an isomorphism of representations T and S. The set of isomorphisms of representations T and S is denoted by $\mathrm{Iso}(T, S)$. The set of automorphisms of the representation T is denoted by $\mathrm{Aut}(T)$.

Lemma 1.6. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations, $A \in \mathrm{Mor}(T, S)$. Then:*
a) $\ker A = \{v \in V \mid Av = 0\} \subset V$ *is an invariant subspace under T;*
b) $\mathrm{im} A = \{Av \mid v \in V\} \subset U$ *is an invariant subspace under S.*

Proposition 1.7. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are irreducible representations, $A \in \mathrm{Mor}(T, S)$. Then either $A = 0$ or $A \in \mathrm{Iso}(T, S)$.*

Proposition 1.8. *(Schur's Lemma) Suppose that $T : G \to \mathrm{GL}(V)$ is an irreducible complex representation, and $A \in \mathrm{End}(T)$. Then $\exists \lambda \in \mathbb{C}$ such that $A = \lambda \mathrm{Id}$.*

Question 1.9. Is the claim of Schur's Lemma still true over $\mathbb{R}$?

Definition 1.10. A vector space V over a field $\mathbb{F}$ with an operation $\cdot : V \times V \to V$ is called an algebra if the following properties hold:

(1) $a(b + c) = ab + ac, \quad \forall a, b, c \in V$;
(2) $(b + c)a = ba + bc, \quad \forall a, b, c \in V$;
(3) $\lambda(ab) = (\lambda a)b = a(\lambda b), \quad \forall a, b \in V, \forall \lambda \in \mathbb{F}$.

An algebra V is called associative if $a(bc) = (ab)c$, $\forall a, b, c \in V$. An associative algebra V is called a division algebra if $(V \setminus \{0\}, \cdot)$ is a group.

Date: September 20, 2025.

Question 1.11. Let $T : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{F}$. Does $\mathrm{End}(T)$ have to be a division algebra over $\mathbb{F}$?

Question 1.12. What are possible finite-dimensional division algebras over $\mathbb{C}$?

Question 1.13. What are possible finite-dimensional division algebras over $\mathbb{R}$?

Exercise 1.14. Construct an irreducible real representation of a finite group such that its algebra of endomorphisms is isomorphic to the division algebra of quaternions. You may pick any finite group you prefer.

2. Complete reducibility of representations.

Definition 2.1. Let $\mathbb{F}$ be a field. Let us denote by $\mathrm{char}\mathbb{F}$ (the characteristic of the field $\mathbb{F}$) the minimal natural number n such that

$$\underbrace{1+1+\cdots+1}_{n \text{ times}} = 0. \tag{2.1}$$

If there is no such natural number we will write $\mathrm{char}\mathbb{F} := 0$.

Proposition 2.2. *(Maschke) Suppose that a group G is finite, $\mathbb{F}$ is a field, V is a vector space over the field $\mathbb{F}$, $T : G \to \mathrm{GL}(V)$ is a representation, and* $\mathrm{char}\mathbb{F}$ *does not divide $|G|$. Then the representation T is completely reducible.*

Proof. Let $U \subset V$ be an invariant subspace. Denote by $W \subset V$ an arbitrary subspace of V such that $U \oplus W = V$. Define $P \in \mathrm{End}(V)$ by

$$P(u) := u, \quad \forall u \in U, \quad P(w) := 0, \quad \forall w \in W. \tag{2.2}$$

Now we will average P over the group G. Let

$$P_0 := \frac{1}{|G|} \sum_{g \in G} T(g) P T(g^{-1}). \tag{2.3}$$

Exercise 2.3. Show that $P_0 \in \mathrm{End}(T)$.

Thus, $\ker(P_0)$ is invariant wrt T.

Exercise 2.4. Show that

$$U \oplus \ker P_0 = V. \tag{2.4}$$

□

Question 2.5. Suppose that $T : G \to \mathrm{GL}(V)$ is a completely reducible representation and V is expressed as a sum of minimal invariant subspaces:

$$V = \bigoplus_{i=1}^{m} V_i \tag{2.5}$$

Is such expression unique up to permutation of summands?

Exercise 2.6. Suppose that the group G is finite and $T : G \to \mathrm{GL}(\mathbb{C}^2)$ is a representation. Suppose $\exists g, h \in G$ such that $T(g)T(h) \neq T(h)T(g)$. Can the representation T be not irreducible?

3. Invariant inner product.

Recall the following result from Linear Algebra.

Proposition 3.1. *Assume $A \in \mathrm{End}(V)$ is a unitary operator, a subspace $U \subset V$ is invariant with respect to A (i. e. we have that $Au \in U$, $\forall u \in U$). Then $U^{\perp} = \{w \in V \,|\, \langle w, u\rangle = 0, \forall u \in U\}$ is also invariant with respect to A.*

We are ready to give an alternative proof of Maschke's Theorem for the complex case.

Proposition 3.2. *Let G be a finite group and let $T : G \to \mathrm{GL}(V)$ be its complex representation. Then there exists an inner product in V such that $T(g)$ is unitary for every $g \in G$.*

In other words, we have shown that in the space of a complex representation of a finite group, we can always define an inner product that is invariant under the action of operators of this representation.

Exercise 3.3. Deduce Maschke's Theorem for the complex case from Proposition 3.2.

4. One-dimensional representations

Consider some examples first.

Exercise 4.1. The quaternion group Q_8 is generated by the elements i, j, ε such that $\varepsilon^2 = e$, $\varepsilon i = i\varepsilon$, $\varepsilon j = j\varepsilon$, $i^2 = j^2 = \varepsilon$, $ij = \varepsilon ji$. Find all its 1-dimensional complex representations.

Exercise 4.2. The dihedral group D_n is generated by a rotation a and a reflection s such that $a^n = s^2 = e$, $sas = a^{-1}$. Find all 1-dimensional complex representations of the group D_n. Note that the answer depends on the parity of n.

For a more general approach let us recall useful constructions from Group Theory.

Definition 4.3. Let G be a group, for any $a, b \in G$ define the commutator $[a, b]$ by $[a, b] := aba^{-1}b^{-1}$. Define the derived subgroup $[G, G]$ as the subgroup generated by the set of elements $\{[a, b] \,|\, a, b \in G\}$.

Proposition 4.4. *Let G be a group, then:*
a) $[G, G]$ is a normal subgroup in G;
b) the quotient group $G/[G, G]$ is abelian;
c) the subgroup $[G, G]$ is a minimal subgroup of G among the normal subgroups H of G with the property that the quotient group G/H is abelian.

Proposition 4.5. *Assume G is a group, $p : G \to G/[G, G]$ is a canonical projection. For every one-dimensional representation $\alpha : G/[G, G] \to \mathbb{F}^*$ we define the one-dimensional representation $\bar{\alpha} : G \to \mathbb{F}^*$ by $\bar{\alpha} = \alpha \circ p$. The map $\alpha \mapsto \bar{\alpha}$ is a bijection between the set of one-dimensional representations of the group $G/[G, G]$ and the set of one-dimensional representations of the group G.*

Exercise 4.6. Show that $[S_n, S_n] = A_n$, where $A_n := sgn^{-1}(1)$.

Exercise 4.7. Find all 1-dimensional complex representations of the group S_n.

Exercise 4.8. Find all 1-dimensional complex representations of the group A_4.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Section 1.1;
- James, Liebeck, Chapters 3 - 5, 7 - 9;
- Vinberg, Sections 1.1 – 1.4, 2.1 – 2.3, 4.1 – 4.3.