LECTURE 3. DECOMPOSITION OF A REPRESENTATION. ONE-DIMENSIONAL REPRESENTATIONS.

Please treat each unproven proposition in this text as an exercise.

1. Existence of a Decomposition of a Completely Reducible Representation

Let us formally define "the smallest blocks" of the representation T in the vector space V.

Definition 1.1. Let $T : G \to \mathrm{GL}(V)$ be a representation of a group G. A subspace $U \subset V$ is called minimal invariant if U is a non-trivial invariant subspace containing no other non-trivial invariant subspace..

One can prove the following useful results about completely reducible representations.

Proposition 1.2. *Every subrepresentation of a completely reducible representation is completely reducible.*

Proposition 1.3. *Suppose that $T : G \to \mathrm{GL}(V)$ is a completely reducible representation and $U \subset V$ is an invariant subspace. Then the quotient representation*

$$T_{V/U} : G \to \mathrm{GL}(V/U) \tag{1.1}$$

is also completely reducible.

Using mathematical induction and Proposition 1.2, we can deduce the existence statement.

Proposition 1.4. *Let $T : G \to \mathrm{GL}(V)$ be a completely reducible representation. Then V can be decomposed into a direct sum of minimal invariant subspaces*

$$V = \bigoplus_{i=1}^{k} V_i. \tag{1.2}$$

2. Uniqueness of a Decomposition of a Representation

We have already seen that the decomposition of V into a direct sum of minimal invariant subspaces is not uniquely determined. Next, we will establish the uniqueness property for the set of corresponding subrepresentations

$$T \cong \bigoplus_{i=1}^{k} T|_{V_i}. \tag{2.1}$$

Proposition 2.1. *Suppose $T : G \to \mathrm{GL}(V)$ is a representation of the group G and*

$$T \cong \bigoplus_{i=1}^{k} T_i \cong \bigoplus_{j=1}^{m} S_j, \tag{2.2}$$

where $T_1, \ldots, T_k, S_1, \ldots, S_m$ are irreducible representations. Then $k = m$ and there exists $\sigma \in S_k$ such that $T_{\sigma(i)} \simeq S_i$, $\forall 1 \leq i \leq k$.

One approach to proving this proposition can be found in Vinberg, Section 3.2, Theorem 4. Next, we will try to expand another proof suggested by Fulton and Harris, Proposition 1.8.

Using the images of the minimal invariant subspaces corresponding to the subrepresentations S_j we can formulate a statement that is equivalent to Proposition 2.1.

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Proposition 2.2. *Suppose $T : G \to \mathrm{GL}(V)$ is a representation of the group G and*

$$V = \bigoplus_{i=1}^{k} V_i = \bigoplus_{j=1}^{m} U_j, \tag{2.3}$$

where each V_i and each U_j are minimal invariant subspaces. Then $k = m$ and $\exists \sigma \in S_k$ such that

$$T|_{V_{\sigma(i)}} \cong T|_{U_i}, \forall 1 \leq i \leq k. \tag{2.4}$$

Proof. We can relabel the subspaces U_i grouping together the ones corresponding to the isomorphic subrepresentations. Then we obtain a set of subspaces

$$\{U_b^{(a)} \mid 1 \leq a \leq r, 1 \leq b \leq t_a\} \text{ for some } r \in \mathbb{Z}_+ \text{ and } t_1, \ldots, t_r \in \mathbb{Z}_+, \tag{2.5}$$

where

$$\left(T|_{U_b^{(a)}} \cong T|_{U_d^{(c)}}\right) \iff (a = c). \tag{2.6}$$

By construction, we have that

$$\{U_b^{(a)} \mid 1 \leq a \leq r, 1 \leq b \leq t_a\} = \{U_j \mid 1 \leq j \leq m\}. \tag{2.7}$$

Thus,

$$V = \bigoplus_{a=1}^{r} \left(\bigoplus_{b=1}^{t_r} U_b^{(a)} \right). \tag{2.8}$$

Denote by $p_b^{(a)}$ the projection operator onto $U_b^{(a)}$ corresponding to this decomposition.

Exercise 2.3. Show that

$$\mathrm{Id} = \sum_{a=1}^{r} \left(\sum_{b=1}^{t_r} p_b^{(a)} \right). \tag{2.9}$$

Therefore, for every $1 \leq i \leq k$ there exist a pair $1 \leq a \leq r, 1 \leq b \leq t_a$ such that

$$p_b^{(a)}(V_i) \neq \{0\}. \tag{2.10}$$

If (2.10) holds, then $p_b^{(a)}$ is an isomorphism between V_i and $U_b^{(a)}$.

Exercise 2.4. Show that

$$\left((p_b^{(a)}(V_i) \neq \{0\}) \text{ and } (p_d^{(c)}(V_i) \neq \{0\}) \right) \implies (a = c). \tag{2.11}$$

Therefore, we can also relabel the subspaces V_i, grouping together those corresponding to isomorphic subrepresentations. So we obtain a set of subspaces

$$\{V_b^{(a)} \mid 1 \leq a \leq r, 1 \leq b \leq l_a\} \text{ for some } l_1, \ldots, l_r \in \mathbb{Z}_+, \tag{2.12}$$

such that

$$\left(T|_{V_b^{(a)}} \cong T|_{V_d^{(c)}}\right) \iff (a = c) \tag{2.13}$$

and

$$T|_{V_b^{(a)}} \cong T|_{U_d^{(a)}}, \quad \forall 1 \leq a \leq r, \forall 1 \leq b \leq l_a, \forall 1 \leq d \leq t_a. \tag{2.14}$$

Again we have that

$$V = \bigoplus_{a=1}^{r} \left(\bigoplus_{b=1}^{l_r} V_b^{(a)} \right). \tag{2.15}$$

Pick an arbitrary $1 \leq a \leq r$. If $c \neq a$, then $p_c^{(d)}(V_b^{(a)}) = \{0\}$, $\forall 1 \leq d \leq t_c$, $\forall 1 \leq b \leq l_a$.

Exercise 2.5. Show that

$$\left(\sum_{i=1}^{t_a} p_i^{(a)} \right) |_{V_b^{(a)}} = \mathrm{Id}|_{V_b^{(a)}}. \tag{2.16}$$

Hence,

$$\bigoplus_{b=1}^{l_r} V_b^{(a)} \subset \bigoplus_{d=1}^{t_r} U_d^{(a)}. \tag{2.17}$$

Combining (2.15) and (2.17), we obtain the equality

$$\bigoplus_{b=1}^{l_r} V_b^{(a)} = \bigoplus_{d=1}^{t_r} U_d^{(a)}. \tag{2.18}$$

From (2.14) it follows that

$$l_a = t_a, \forall 1 \leq a \leq r, \tag{2.19}$$

which finishes the proof. □

Definition 2.6. Let G be a finite group. Denote by $\widehat{G}$ the set of equivalence classes of irreducible complex representations of the group G. Allowing a slight abuse of notation, we will also denote by $\widehat{G}$ an arbitrary set that contains one representative from each equivalence class of irreducible complex representations of the group G.

Definition 2.7. For a representation $T : G \to \mathrm{GL}(V)$ and a natural number m, we will denote:

$$mT := \underbrace{T \oplus T \oplus \cdots \oplus T}_{n \text{ times}}. \tag{2.20}$$

From Proposition 1.4, it follows that every real or complex representation T of a finite group G is isomorphic to the sum of irreducible representations with some multiplicities:

$$T \cong \bigoplus_{S \in \widehat{G}} m_s S \tag{2.21}$$

By Proposition 2.1, those multiplicities are uniquely determined by T.

Exercise 2.8. Suppose that T is a complex representation of the group G and

$$T \cong \bigoplus_{S \in \widehat{G}} m_s S. \tag{2.22}$$

Then

$$m_s = \dim(\mathrm{Mor}(S, T)) = \dim(\mathrm{Mor}(T, S)), \quad \forall S \in \widehat{G}. \tag{2.23}$$

James and Liebeck use (2.23) to prove Proposition 2.1 when $\mathbb{F} = \mathbb{C}$, see James, Liebeck, Section 11.

3. One-dimensional representations

First, consider some examples.

Exercise 3.1. The quaternion group Q_8 is generated by the elements i, j, ε such that $\varepsilon^2 = e$, $\varepsilon i = i\varepsilon$, $\varepsilon j = j\varepsilon$, $i^2 = j^2 = \varepsilon$, $ij = \varepsilon ji$. Find all its 1-dimensional complex representations.

Exercise 3.2. The dihedral group D_n is generated by a rotation a and a reflection s such that $a^n = s^2 = e$, $sas = a^{-1}$. Find all 1-dimensional complex representations of the group D_n. Note that the answer depends on the parity of n.

For a more general approach, let us recall useful constructions from Group Theory.

Definition 3.3. Let G be a group. For any $a, b \in G$, define the commutator $[a, b]$ by $[a, b] := aba^{-1}b^{-1}$. Define the derived subgroup $[G, G]$ as the subgroup generated by the set of elements $\{[a, b] \,|\, a, b \in G\}$.

Proposition 3.4. *Let G be a group. Then:*
a) $[G,G]$ is a normal subgroup in G;
b) the quotient group $G/[G,G]$ is abelian;
c) the subgroup $[G,G]$ is a minimal subgroup of G among the normal subgroups H of G with the property that the quotient group G/H is abelian.

Proposition 3.5. *Suppose G is a group, $p : G \to G/[G,G]$ is a canonical projection. For every one-dimensional representation $\alpha : G/[G,G] \to \mathbb{F}^*$ we define the one-dimensional representation $\bar{\alpha} : G \to \mathbb{F}^*$ by $\bar{\alpha} = \alpha \circ p$. The map $\alpha \mapsto \bar{\alpha}$ is a bijection between the set of one-dimensional representations of the group $G/[G,G]$ and the set of one-dimensional representations of the group G.*

Exercise 3.6. Prove that $[S_n, S_n] = A_n$, where $A_n := sgn^{-1}(1)$.

Exercise 3.7. Find all 1-dimensional complex representations of the group S_n.

Exercise 3.8. Find all 1-dimensional complex representations of the group A_4.

4. Dual Representation.

Definition 4.1. Let $T : G \to \mathrm{GL}(V)$ be a representation over a field F. Define its dual representation $T^* : G \to \mathrm{GL}(V^*)$ by

$$\boxed{T^*(g) = (T(g^{-1}))^*.} \tag{4.1}$$

Exercise 4.2. Show that T^* is indeed a representation of the group G.

Exercise 4.3. Define $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^3)$ as

$$T(n) = \begin{pmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that T is a representation. Find the matrix of $T^*(n)$ in the dual basis.

Exercise 4.4. Let $T : G \to \mathrm{GL}(V)$ be a representation. Is it always true that the representations T and $(T^*)^*$ are isomorphic?

Exercise 4.5. Let T be an irreducible representation of a group G. Is it possible that the representation T^* is not irreducible?

5. Tensor Product of Representations.

Definition 5.1. Let $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ be representations over a field $\mathbb{F}$. We define the tensor product of representations T and S $T \otimes S : G \to \mathrm{GL}(V \otimes U)$ by the following equality:

$$\boxed{(T \otimes S)(g) := (T(g)) \otimes (S(g)).} \tag{5.1}$$

Exercise 5.2. Let $T_1, \ldots, T_n$ be the set of all pairwise non-isomorphic one-dimensional complex representations of G. Prove that this set forms a group with respect the elementwise multiplication. Describe that group for $G = \mathbb{Z}_n$, S_n.

Exercise 5.3. Let T be a representation of a finite group G. Is it true that the representation $T \otimes T^*$ of the group G always has a one-dimensional invariant subspace?

Definition 5.4. Let $T : G \to \mathrm{GL}(V)$ be a representation. Define the symmetric square S^2T : $G \to \mathrm{GL}(S^2(V))$ and the exterior square $\Lambda^2 T : G \to \mathrm{GL}(\Lambda^2(V)$ of the representation T by

$$\boxed{(S^2T)(g) := S^2(T(g)), \quad (\Lambda^2 T)(g) := \Lambda^2(T(g)).} \tag{5.2}$$

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Section 1.2;
- James, Liebeck, Chapters 8, 9, 11;
- Vinberg, Sections 1.5, 3.1-3.4, 3.9.