LECTURE 4. CHARACTER OF A REPRESENTATION.

Please treat each unproven proposition in this text as an exercise.

1. Complexification.

Definition 1.1. Let V be a real vector space. Let us consider the field $\mathbb{C}$ as a vector space over $\mathbb{R}$. The tensor product $\mathbb{C}\otimes V$ is also a real vector space. We will introduce a complex vector space structure on $\mathbb{C}\otimes V$ by setting

$$\lambda\cdot(z\otimes v):=(\lambda z\otimes v),\quad \forall\lambda,z\in\mathbb{C},\forall v\in V. \tag{1.1}$$

Here, we define multiplication by a fixed λ on pairs $\mathbb{C}\times V$. Note that this map is bilinear; we then extend this map to the tensor product $\mathbb{C}\otimes V$ using its universal property. We also define addition to be the same as in the tensor product $\mathbb{C}\otimes V$. The resulting complex vector space is called the complexification of the real vector space V and is denoted by $V^{\mathbb{C}}$. Also define

$$V_0^{\mathbb{C}}:=\{1\otimes v\mid v\in V\}. \tag{1.2}$$

For every subspace U of the vector space V, denote by $U^{\mathbb{C}}$ the following subspace of $V^{\mathbb{C}}$:

$$U^{\mathbb{C}}:=\{z\otimes u\mid z\in\mathbb{C},u\in U\} \tag{1.3}$$

Exercise 1.2. Let V be a real vector space. Show that $\forall v\in V^{\mathbb{C}}\ \exists u,w\in V$ such that

$$v=1\otimes u+i\otimes w. \tag{1.4}$$

Exercise 1.3. Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V<\infty$, then the dimension of $V^{\mathbb{C}}$ over C equals $\dim V$. What can be said about $V_0^{\mathbb{C}}$ and its dimension?

Definition 1.4. Suppose that V is a real vector space and $A\in\mathrm{End}(V)$. We define the complexification $A^{\mathbb{C}}\in\mathrm{End}(V^{\mathbb{C}})$ of the linear operator A by

$$A^{\mathbb{C}}(z\otimes v):=z\otimes(Av). \tag{1.5}$$

Note that

$$\forall v\in V_0^{\mathbb{C}}\quad A^{\mathbb{C}}v\in V_0^{\mathbb{C}}, \tag{1.6}$$

and the restriction of $A^{\mathbb{C}}$ to $V_0^{\mathbb{C}}$ coincides with A. Thus, we can view $A^{\mathbb{C}}$ as an extension of the operator A defined on $V_0^{\mathbb{C}}$. This viewpoint will be extensively used in the next section.

Example 1.5. Suppose that V is a real vector space, $\alpha=\{v_1,\dots,v_n\}$ is a basis of V, and $A\in\mathrm{End}(V)$. Then the set $\beta=\{1\otimes v_1,\dots,1\otimes v_n\}$ is a basis of $V^{\mathbb{C}}$, and the corresponding matrices coincide:

$$[A^{\mathbb{C}}]_\beta=[A]_\alpha. \tag{1.7}$$

Proposition 1.6. *Suppose that V is a real vector space and $A\in\mathrm{End}(V)$. Then the following equality holds for the characteristic polynomials*

$$\chi_{A^{\mathbb{C}}}(x)=\chi_A(x),\quad \forall x\in\mathbb{R}. \tag{1.8}$$

Note that $\chi_A\in\mathbb{R}[x]$. Thus, Proposition 1.6 states that the characteristic polynomial of the linear operator $A^{\mathbb{C}}$ has real coefficients and is equal to χ_A.

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Example 1.7. Suppose that V is a complex vector space, $\gamma = \{v_1, \dots, v_n\}$ is a basis of V, $D \in \mathrm{End}(V)$, and

$$[D]_\gamma = (a_k^l + ib_k^l)_{1 \le k,l \le n} = A + iB, \quad A, B \in \mathrm{Mat}(n \times n, \mathbb{R}). \tag{1.9}$$

Now let us consider V as a vector space over $\mathbb{R}$, denoted by $V^{\mathbb{R}}$ to avoid confusion. Then the map D is a linear operator of $V^{\mathbb{R}}$; denote it by $D^{\mathbb{R}}$. Then $\delta = \{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ is a basis of $V^{\mathbb{R}}$. We get that

$$[D^{\mathbb{R}}]_\delta = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{1.10}$$

Proposition 1.8. *Let V be a finite-dimensional real vector space. Define* Conj $\in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *by*

$$\mathrm{Conj}(z \otimes v) := \overline{z} \otimes v. \tag{1.11}$$

Then:

a) $\mathrm{Conj}^2 = \mathrm{Id}$*;*

b) $\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ A^{\mathbb{C}} = A^{\mathbb{C}} \circ \mathrm{Conj}$*;*

c) We get that $\frac{1}{2}(\mathrm{Id} + \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and it is a projection operator onto* $V_0^{\mathbb{C}}$*;*

d) We get that $\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and*

$$\mathrm{im}\left(\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})\right) = V_0^{\mathbb{C}}; \tag{1.12}$$

e)

$$\mathrm{im}\left(\frac{1}{2}(\mathrm{Id} + \mathrm{Conj})\right) \oplus \mathrm{im}\left(\frac{1}{2}(\mathrm{Id} - \mathrm{Conj})\right) = (V^{\mathbb{C}})^{\mathbb{R}}. \tag{1.13}$$

Proposition 1.9. *Suppose that V is a real finite-dimensional vector space and $A \in \mathrm{End}(V)$. Denote by V^λ a generalized eigenspace of $A^{\mathbb{C}}$. Then* $\mathrm{Conj}(V^\lambda) = V^{\overline{\lambda}}$.

Proposition 1.10. *Suppose that V is a real finite-dimensional vector space and W is a subspace of $V^{\mathbb{C}}$. Then the following conditions are equivalent:*

$$(\mathrm{Conj}(W) = W) \quad \Longleftrightarrow \quad (\textit{There exists a subspace } U \subset V \textit{ such that } W = U^{\mathbb{C}}). \tag{1.14}$$

Definition 1.11. For a representation $T : G \to \mathrm{GL}(V)$ we define its complexification $T^{\mathbb{C}} : G \to \mathrm{GL}(V^{\mathbb{C}})$ by

$$T^{\mathbb{C}}(g) := (T(g))^{\mathbb{C}}. \tag{1.15}$$

Proposition 1.12. *Let $T : G \to \mathrm{GL}(V)$ be an irreducible real representation of odd dimension. Then its complexification $T^{\mathbb{C}}$ is also irreducible.*

Exercise 1.13. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is an irreducible complex representation.

Exercise 1.14. Find an example of an irreducible real representation T such that $\dim T$ is even and $T^{\mathbb{C}}$ is not irreducible.

2. Geometric constructions of representations

Exercise 2.1. Consider an equilateral triangle on the plane with the center at the point $(0, 0)$. The symmetric group S_3 acts on this triangle by permutations of its vertices. Show that the corresponding 2-dimensional representation T of the group S_3 over $\mathbb{R}$ is irreducible. Now consider these real-valued matrices $T(\sigma)$ for $\sigma \in S_3$ as complex-valued. Show that the corresponding complex representation $T^{\mathbb{C}}$ (complexification) of the group S_3 over $\mathbb{C}$ is also irreducible.

Exercise 2.2. Consider a regular tetrahedron in the Euclidean space $\mathbb{R}^3$ with the center at the origin. Denote by $O(T)$ the set of isometries of $\mathbb{R}^3$ preserving the tetrahedron. Also put $SO(T) = \{A \in O(T) \mid \det(A) > 0\}$. Show that

$$O(T) \simeq S_4, \quad SO(T) \simeq A_4. \tag{2.1}$$

Exercise 2.3. Consider a cube in the Euclidean space $\mathbb{R}^3$ with the center at the origin. Denote by $O(C)$ the set of isometries of $\mathbb{R}^3$ preserving the cube. Also put $SO(C) = \{A \in O(C) \mid \det(A) > 0\}$. Show that

$$O(C) \simeq S_4 \times \mathbb{Z}_2, \quad SO(C) \simeq S_4. \tag{2.2}$$

Exercise 2.4. Show that the three-dimensional real representations of the group A_4 determined by the motions of the regular tetrahedron and the cube are irreducible.

So we get also two three-dimensional irreducible real representations of the group S_4. One can show that because their dimensions are odd, the complexifications of those representations are also irreducible.

Exercise 2.5. Is the representation of the group S_4 in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?

Exercise 2.6. Is the representation of the group A_4 in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?

Exercise 2.7. Let $k \in \mathbb{Z}$. Define a 2-dimensional complex representation T_k of the group D_n as

$$T_k(a) = \begin{pmatrix} \cos\frac{2\pi k}{n} & \sin\frac{2\pi k}{n} \\ -\sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{pmatrix} \qquad T_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Check that the representation T_k is well-defined for all integer k. When is the representation T_k irreducible?

3. Character of a representation.

Definition 3.1. Let $T : G \to \mathrm{GL}(V)$ be a representation over a field $\mathbb{F}$. Define the character $\chi_T : G \to \mathbb{F}$ of the representation T by

$$\chi_T(g) := \mathrm{tr}(T(g)), \quad g \in G. \tag{3.1}$$

Proposition 3.2. *a)* $\chi_T(e) = \dim(T)$;
b) $\chi_T(hgh^{-1}) = \chi_T(g)$, $\forall g, h \in G$;
c) χ_T *is constant on the conjugacy classes of the group* G;
d) $\chi_{T\oplus S} = \chi_T + \chi_S$;
e) $\chi_{T\otimes S} = \chi_T \cdot \chi_S$.

Exercise 3.3. Let $T : G \to \mathrm{GL}(V)$ be a complex representation of a finite group G. Show that

$$|\chi_T(g)| \leq \dim T, \quad \forall g \in G. \tag{3.2}$$

Exercise 3.4. Assume that $T : G \to \mathrm{GL}(V)$ is a complex representation. Can we express the character $\chi_{\Lambda^2 T}$ using χ_T ?

From Maschke's Theorem, we deduce that every real or complex representation T of a finite group G is isomorphic to the sum of irreducible representations with some multiplicities:

$$T \cong \bigoplus_{S \in \widehat{G}} m_s S. \tag{3.3}$$

Those multiplicities are uniquely determined by T.

Taking the character of both sides, we get the equality

$$\chi_T = \sum_{S \in \widehat{G}} m_S \chi_S. \tag{3.4}$$

4. Orthogonality relations.

Definition 4.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the set of functions from G to $\mathbb{F}$. The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. For the case $\mathbb{F} = \mathbb{C}$ we introduce an inner product on $\mathbb{C}[G]$ via the formula

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{x \in G} f_1(x)\overline{f_2(x)}. \tag{4.1}$$

Also, define

$$Z(\mathbb{C}[G]) := \{f \in \mathbb{C}[G] \,|\, f(gxg^{-1}) = f(x), \forall x, g \in G\}. \tag{4.2}$$

$Z(\mathbb{C}[G])$ is called the subspace of class functions on G.

The next lectures will be devoted to the proof of the following remarkable result.

Proposition 4.2. *Let G be a finite group. The set $\{\chi_T \,|\, T \in \widehat{G}\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

There are several corollaries of that claim that we are going to use.

Proposition 4.3. *Suppose that T and S are irreducible complex representations of a finite group G. Then*

$$\langle \chi_T, \chi_S \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_T(g)\overline{\chi_S(g)} = \begin{cases} 0, & \text{if } T \text{ is not isomorphic to } S, \\ 1, & \text{otherwise.} \end{cases} \tag{4.3}$$

Proposition 4.4. *Suppose that T and S are irreducible complex representations of a finite group G. If $\chi_T = \chi_S$, then the representations T and S are isomorphic.*

Proposition 4.5. *Let T be a complex representation of a finite group G. If $\langle \chi_T, \chi_T \rangle = 1$ then T is irreducible.*

Proposition 4.6. *Assume that G is a finite group, $x, y \in G$. Then*

$$\sum_{T \in \widehat{G}} \chi_T(x)\overline{\chi_T(y)} = \begin{cases} \frac{|G|}{|C|}, & \text{if } x \text{ and } y \text{ belong to the same conjugacy class } C, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Proposition 4.7. *Let G be a finite group. Then the following identity holds*

$$\sum_{T \in \widehat{G}} (\dim T)^2 = |G|. \tag{4.5}$$

5. Group Algebra.

Definition 5.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the following set of formal linear combinations

$$\mathbb{F}[G] = \{\sum_{g \in C} a_g g \,|\, a_g \in \mathbb{F}, \forall g \in G\}. \tag{5.1}$$

The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. Using distributivity, we can extend the operation of multiplication from the group G to $\mathbb{F}[G]$. $\mathbb{F}[G]$ is called a group algebra. It is indeed an unital associative algebra over the field $\mathbb{F}$.

Question 5.2. As a vector space, the group algebra is isomorphic to the space of functions on the group introduced in the previous lecture. How can we interpret the multiplication in the group algebra in terms of functions?

Recall the definition from the previous lecture.

Definition 5.3. Let G be a finite group. Define the subspace $Z(\mathbb{C}[G])$ of (complex) class functions on G by

$$Z(\mathbb{C}[G]) := \{f \in \mathbb{C}[G] \mid f(gxg^{-1}) = f(x), \forall x, g \in G\}. \tag{5.2}$$

Question 5.4. How can we describe the subspace of class functions in terms of the group algebra?

It is natural to construct an extension of a representation of a group to its group algebra.

Exercise 5.5. For a representation $T : G \to \mathrm{GL}(V)$ of a finite group G over a field $\mathbb{F}$ we can define a unique linear extension of the representation T to the group algebra $\mathbb{F}[G]$, denote it by $\overline{T}$. By definition, $\overline{T}$ maps $\mathbb{F}[G]$ to $\mathrm{End}(V)$. Show that if $\overline{T}(\mathbb{F}[G]) = \mathrm{End}(V)$ then T is irreducible.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 2.1 – 2.4;
- James, Liebeck, Chapters 10-16, 18,19;
- Vinberg, Sections 3-6.