LECTURE 5. CHARACTER OF A REPRESENTATION.

Please treat each unproven statement in this text, except Proposition 2.2, as an exercise.

1. Character of a representation.

Definition 1.1. Let $T : G \to \mathrm{GL}(V)$ be a representation over a field $\mathbb{F}$. Define the character $\chi_T : G \to \mathbb{F}$ of the representation T by

$$\chi_T(g) := \mathrm{tr}(T(g)), \quad g \in G. \tag{1.1}$$

Proposition 1.2. *a)* $\chi_T(e) = \dim(T)$;
b) $\chi_T(hgh^{-1}) = \chi_T(g)$, $\forall g, h \in G$;
c) χ_T *is constant on the conjugacy classes of the group* G;
d) $\chi_{T\oplus S} = \chi_T + \chi_S$;
e) $\chi_{T\otimes S} = \chi_T \cdot \chi_S$.

Exercise 1.3. Let $T : G \to \mathrm{GL}(V)$ be a complex representation of a finite group G. Show that

$$|\chi_T(g)| \leq \dim T, \quad \forall g \in G. \tag{1.2}$$

Exercise 1.4. Suppose that $T : G \to \mathrm{GL}(V)$ is a complex representation. Can we express the character $\chi_{\Lambda^2 T}$ using χ_T?

We deduce from Maschke's Theorem that every real or complex representation T of a finite group G is isomorphic to a sum of irreducible representations with some multiplicities:

$$T \cong \bigoplus_{S\in\widehat{G}} m_s S. \tag{1.3}$$

The multiplicities are uniquely determined by T.

Taking the character of both sides, we get the equality

$$\chi_T = \sum_{S\in\widehat{G}} m_S \chi_S. \tag{1.4}$$

2. Orthogonality relations.

Definition 2.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the set of functions from G to $\mathbb{F}$. The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. For the case $\mathbb{F} = \mathbb{C}$ we introduce an inner product on $\mathbb{C}[G]$ via the formula

$$\langle f_1, f_2\rangle := \frac{1}{|G|}\sum_{x\in G} f_1(x)\overline{f_2(x)}. \tag{2.1}$$

Also, define

$$Z(\mathbb{C}[G]) := \{f \in \mathbb{C}[G] \,|\, f(gxg^{-1}) = f(x), \forall x, g \in G\}. \tag{2.2}$$

$Z(\mathbb{C}[G])$ is called the subspace of class functions on G.

The next lecture will be devoted to the proof of the following remarkable result.

Proposition 2.2. *Let* G *be a finite group. The set* $\{\chi_T \,|\, T \in \widehat{G}\}$ *is an orthonormal basis of* $Z(\mathbb{C}[G])$.

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We will use several corollaries of that claim.

Proposition 2.3. *Suppose that T and S are irreducible complex representations of a finite group G. Then*

$$\langle \chi_T, \chi_S \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_T(g) \overline{\chi_S(g)} = \begin{cases} 0, & \text{if } T \text{ is not isomorphic to } S, \\ 1, & \text{otherwise.} \end{cases} \tag{2.3}$$

Proposition 2.4. *Suppose that T and S are irreducible complex representations of a finite group G. If $\chi_T = \chi_S$, then the representations T and S are isomorphic.*

Proposition 2.5. *Let T be a complex representation of a finite group G. If $\langle \chi_T, \chi_T \rangle = 1$, then T is irreducible.*

Proposition 2.6. *Suppose that G is a finite group and that $x, y \in G$. Then*

$$\sum_{T \in \widehat{G}} \chi_T(x) \overline{\chi_T(y)} = \begin{cases} \frac{|G|}{|C|}, & \text{if } x \text{ and } y \text{ belong to the same conjugacy class } C, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Proposition 2.7. *Let G be a finite group. Then the following identity holds*

$$\sum_{T \in \widehat{G}} (\dim T)^2 = |G|. \tag{2.5}$$

Exercise 2.8. Describe all irreducible characters of the group S_3.

Exercise 2.9. Find the multiplicities of the decomposition of every tensor product of two irreducible complex representations (they may be equal) of the group S_3 into the sum of irreducible ones.

3. Group Algebra.

Definition 3.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the following set of formal linear combinations

$$\mathbb{F}[G] = \{ \sum_{g \in C} a_g g \,|\, a_g \in \mathbb{F}, \forall g \in G \}. \tag{3.1}$$

The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. Using distributivity, we can extend the operation of multiplication from the group G to $\mathbb{F}[G]$. $\mathbb{F}[G]$ is called a group algebra. It is indeed a unital associative algebra over the field $\mathbb{F}$.

Question 3.2. As a vector space, the group algebra is isomorphic to the space of functions on the group introduced in the previous lecture. How can we interpret the multiplication in the group algebra in terms of functions?

Recall the definition from the previous lecture.

Definition 3.3. Let G be a finite group. Define the subspace $Z(\mathbb{C}[G])$ of (complex) class functions on G by

$$Z(\mathbb{C}[G]) := \{ f \in \mathbb{C}[G] \,|\, f(gxg^{-1}) = f(x), \forall x, g \in G \}. \tag{3.2}$$

Question 3.4. How can we describe the subspace of class functions in terms of the group algebra?

It is natural to construct an extension of a representation of a group to its group algebra.

Exercise 3.5. For a representation $T : G \to \mathrm{GL}(V)$ of a finite group G over a field $\mathbb{F}$ there is a unique linear extension of the representation T to the group algebra $\mathbb{F}[G]$, denote it by $\overline{T}$. By definition, $\overline{T}$ maps $\mathbb{F}[G]$ to $\mathrm{End}(V)$. Show that if $\overline{T}(\mathbb{F}[G]) = \mathrm{End}(V)$ then T is irreducible.

4. Regular representations.

Let us introduce three important representations of an arbitrary finite group G.

Definition 4.1. Let G be a finite group. Define the representations $\mathrm{LReg} : G \to \mathrm{GL}(\mathbb{C}[G])$ and $\mathrm{RReg} : G \to \mathrm{GL}(\mathbb{C}[G])$ by the following formulas:

$$(\mathrm{LReg}(g)f)(x) := f(g^{-1}x), \quad \forall g, x \in G, \forall f \in \mathbb{C}[G], \tag{4.1}$$

$$(\mathrm{RReg}(g)f)(x) := f(xg), \quad \forall g, x \in G, \forall f \in \mathbb{C}[G]. \tag{4.2}$$

Also define the representation $\mathrm{BiReg} : G \times G \to \mathrm{GL}(\mathbb{C}[G])$ by

$$(\mathrm{BiReg}(g,h)f)(x) := f(g^{-1}xh), \quad \forall g, h, x \in G, \forall f \in \mathbb{C}[G]; \tag{4.3}$$

These representations are called the left regular, the right regular, and the biregular representations respectively.

Exercise 4.2. Find the characters of those three representations.

Exercise 4.3. Find the multiplicities in the decomposition of the representation LReg into the sum of irreducible complex representations of the group G.

5. External tensor product of representations

Definition 5.1. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(V)$ are representations over the same field. Define an external tensor product $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ of representations T and S by

$$(T \otimes S)(g,h) := T(g) \otimes S(h), \quad \forall g \in G, \forall h \in H. \tag{5.1}$$

It might be confusing that the same symbol is used to denote both the external tensor product and the usual tensor product of representations of the same group.

Proposition 5.2. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(U)$ are representations over the same field. Then the representation $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ is irreducible if and only if both T and S are irreducible.*

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 2.1 – 2.4;
- James, Liebeck, Chapters 10-16, 18,19;
- Vinberg, Sections 5, 6.