LECTURE 6. ORTHOGONALITY RELATIONS.

Please treat each unproven statement in this text as an exercise.

1. Group Algebra.

Definition 1.1. Suppose that $\mathbb{F}$ is a field and G is a finite group. Denote by $\mathbb{F}[G]$ the following set of formal linear combinations

$$\mathbb{F}[G] = \{\sum_{g\in C} a_g g \,|\, a_g \in \mathbb{F}, \forall g \in G\}. \tag{1.1}$$

The set $\mathbb{F}[G]$ has a natural structure of a vector space over $\mathbb{F}$. Using distributivity, we can extend the operation of multiplication from the group G to $\mathbb{F}[G]$. $\mathbb{F}[G]$ is called a group algebra. It is a unital associative algebra over the field $\mathbb{F}$.

Question 1.2. As a vector space, the group algebra is isomorphic to the space of functions on the group introduced in the previous lecture. How can we interpret multiplication in the group algebra in terms of functions?

Recall the definition from the previous lecture.

Definition 1.3. Let G be a finite group. Define the subspace $Z(\mathbb{C}[G])$ of (complex-valued) class functions on G by

$$Z(\mathbb{C}[G]) := \{f \in \mathbb{C}[G] \,|\, f(gxg^{-1}) = f(x), \forall x, g \in G\}. \tag{1.2}$$

Question 1.4. How can we describe the subspace of class functions in terms of the group algebra?

It is natural to construct an extension of a representation of a group to its group algebra.

Exercise 1.5. For a representation $T : G \to \mathrm{GL}(V)$ of a finite group G over a field $\mathbb{F}$ there is a unique linear extension of the representation T to the group algebra $\mathbb{F}[G]$, denote it by $\overline{T}$. By definition, $\overline{T}$ maps $\mathbb{F}[G]$ to $\mathrm{End}(V)$.

Exercise 1.6. Give a definition of a representation of an algebra. Also give definitions of invariant subspaces of such representations. For a representation $T : G \to \mathrm{GL}(V)$ of a group G over a field $\mathbb{F}$ show that T is irreducible if and only if the corresponding representation $\overline{T}$ of its group algebra $\mathbb{F}[G]$ is irreducible.

Exercise 1.7. Show that if $\overline{T}(\mathbb{F}[G]) = \mathrm{End}(V)$, then T is irreducible.

2. Orthogonality of matrix elements.

Recall the orthogonality relations for irreducible characters.

Proposition 2.1. *Suppose that G is a finite group, T and S are irreducible complex representations of G. Then*

$$\langle \chi_T, \chi_S \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_T(g)\overline{\chi_S(g)} = \begin{cases} 0, & \text{if } T \text{ is not isomorphic to } S, \\ 1, & \text{otherwise.} \end{cases} \tag{2.1}$$

We are going to prove a stronger claim.

Date: October 18, 2025.

Proposition 2.2 (Orthogonality of matrix elements)**.** *Suppose that G is a finite group, $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(W)$ are irreducible complex representations of G, $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ are invariant inner products in the vector spaces V and W respectively, $x, y \in V$, $z, t \in W$. Then*

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V \overline{\langle S(g)z, t\rangle_W} =$$

$$= \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ \frac{1}{\dim T}\langle x, z\rangle_V \overline{\langle y, t\rangle_V}, & \text{otherwise (here we identify } T \text{ with } S, \text{ and } V \text{ with } W). \end{cases} \tag{2.2}$$

Proof. We are going to use the following lemma.

Lemma 2.3. *Suppose V and W are complex inner product vector spaces, $f : V \times W \to \mathbb{C}$ is a sesquilinear function, i.e. f is linear with respect to the first variable and conjugate linear with respect to the second variable. Then $\exists A \in \mathrm{Hom}(V, W)$ such that*

$$f(x, z) = \langle Ax, z\rangle_W. \tag{2.3}$$

For any $y \in V, t \in W$ define

$$f_1(x, z) = \frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V \overline{\langle S(g)z, t\rangle_W} \tag{2.4}$$

Applying Lemma 2.3 we get that $\exists A \in \mathrm{Hom}(V, W)$ such that

$$f_1(x, z) = \langle Ax, z\rangle_W. \tag{2.5}$$

Then $\forall h \in G$, $\forall x \in V$, $\forall z \in W$ we obtain

$$\langle AT(h)x, z\rangle_W = f_1(T(h)x, z) = \frac{1}{|G|}\sum_{g\in G}\langle T(g)T(h)x, y\rangle_V \overline{\langle S(g)z, t\rangle_W} =$$

$$= \frac{1}{|G|}\sum_{g'\in G}\langle T(g')x, y\rangle_V \overline{\langle S(g')S(h^{-1})z, t\rangle_W} = f_1(x, S(h^{-1})z) =$$

$$= \langle Ax, S(h^{-1})z\rangle_W = \langle S(h)Ax, z\rangle_W \tag{2.6}$$

Thus,

$$AT(h) = S(h)A, \quad \forall h \in G. \tag{2.7}$$

Therefore, by Schur's lemma,

$$A = \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ \lambda \cdot Id, & \text{otherwise.} \end{cases} \tag{2.8}$$

Note that the linear map A and, hence, the number λ depend conjugate linearly on y and linearly on t. As a result, we obtain that

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V \overline{\langle S(g)z, t\rangle_W} = \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ f_2(t, y)\langle x, z\rangle_V, & \text{otherwise,} \end{cases} \tag{2.9}$$

where f_2 is a sesquilinear function. Proposition 2.2 has been proven when T is not equivalent to S.

Now let us consider the case when $T = S$, $V = W$. From the equality (2.9) and applying Lemma 2.3 once again we obtain that

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x, y\rangle_V \overline{\langle T(g)z, t\rangle_V} = c_T\langle x, z\rangle_V \overline{\langle y, t\rangle_V}, \tag{2.10}$$

where c_T is a complex number that depends on T.

Let us find c_T. Suppose that $e_1, e_2, \ldots, e_{\dim T}$ is an orthonormal basis of V. We know that $T(g)$ is a unitary operator for $\forall g \in G$. Therefore,

$$\sum_{i=1}^{\dim T} |\langle T(g)e_1, e_i\rangle|^2 = 1 \tag{2.11}$$

Let us define $x = z = e_1$, $y = t = e_i$ for $1 \le i \le \dim T$ in the equality (2.10) and sum those $\dim T$ equalities. Then using (2.11) we obtain that $\dim T \cdot c_T = 1$. The proposition has been proven. □

Exercise 2.4. Show that Proposition 2.1 follows from Proposition 2.2.

From Proposition 2.1 we obtain that irreducible characters form a set of pairwise orthogonal vectors in the finite-dimensional vector space $Z(\mathbb{C}[G])$. Therefore, we have proved that if the group G is finite then the set $\widehat{G}$ is also finite.

3. Regular Representations.

Let us introduce three important representations of an arbitrary finite group G.

Definition 3.1. Let G be a finite group. Define the representations $\mathrm{LReg} : G \to \mathrm{GL}(\mathbb{C}[G])$ and $\mathrm{RReg} : G \to \mathrm{GL}(\mathbb{C}[G])$ by the following formulas:

$$(\mathrm{LReg}(g)f)(x) := f(g^{-1}x), \quad \forall g, x \in G, \forall f \in \mathbb{C}[G], \tag{3.1}$$

$$(\mathrm{RReg}(g)f)(x) := f(xg), \quad \forall g, x \in G, \forall f \in \mathbb{C}[G]. \tag{3.2}$$

Also, define the representation $\mathrm{BiReg} : G \times G \to \mathrm{GL}(\mathbb{C}[G])$ by

$$(\mathrm{BiReg}(g,h)f)(x) := f(g^{-1}xh), \quad \forall g, h, x \in G, \forall f \in \mathbb{C}[G]; \tag{3.3}$$

These representations are called the left regular, the right regular, and the biregular representations respectively.

Exercise 3.2. Find the characters of those three representations.

Exercise 3.3. Find the multiplicities in the decomposition of the representation LReg into the sum of irreducible complex representations of the group G.

4. Algebraic isomorphism of a complex group algebra.

Proposition 4.1. *Let G be a finite group. The set $\{\chi_T \,|\, T \in \widehat{G}\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

We are going to deduce it from the following beautiful construction.

Proposition 4.2. *Suppose that G is a finite group. Then the set $\widehat{G}$ is finite. Denote $k = |\widehat{G}|$ and $\widehat{G} = \{T_i : G \to \mathrm{GL}(V_i) \,|\, 1 \le i \le k\}$. Define*

$$V = \bigoplus_{i=1}^{k} V_i \tag{4.1}$$

Define $B \in \mathrm{Hom}(\mathbb{C}[G], \mathrm{End}(V))$ by

$$B(\sum_{g\in G} a_g g)(v_1, v_2, \ldots, v_k) = (\sum_{g\in G} a_g(T_1(g)(v_1)), \sum_{g\in G} a_g(T_2(g)(v_2)), \ldots, \sum_{g\in G} a_g(T_k(g)(v_k))) \tag{4.2}$$

Then B is an isomorphism of $\mathbb{C}[G]$ and $\prod_{i=1}^{k} \mathrm{End}(V_i)$.

Proof. We need to show that the map B is injective and surjective.

First, let us prove the surjectivity. Consider an arbitrary sequence $D = (D_1, \dots, D_k), \quad D_i \in \mathrm{End}(V_i)$. It is sufficient to show that we may find a preimage of the sequence D when

$$D_t = 0, \forall t \neq s \text{ and } D_s = E_i^j \text{ where } 1 \leq s \leq k,\ 1 \leq i, j \leq \dim T_s. \tag{4.3}$$

Here E_i^j denotes the matrix unit with a 1 in the i-th row and the j-th column. Without loss of generality, we may assume that $s = 1$. Denote by $\langle \cdot, \cdot \rangle_{V_1}$ an invariant inner product in the vector space V_1. Let $e_1, \dots, e_{\dim T_1}$ be the orthonormal basis relative to which we consider E_i^j. The matrix unit E_i^j is determined by the condition

$$\langle E_i^j e_k, e_l \rangle_{V_1} = \begin{cases} 1, & \text{if } k = j,\ i = l, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Define

$$x_{i,j} := \frac{\dim T_1}{|G|} \sum_{g \in G} \overline{\langle T_1(g) e_j, e_i \rangle_{V_1}}\, g. \tag{4.5}$$

From Proposition 2.2 we get that if $m \neq 1$ then

$$\langle T_m(x_{i,j}) u, w \rangle_{V_m} = \frac{\dim T_1}{|G|} \sum_{g \in G} \langle T_m(g) u, w \rangle_{V_m} \overline{\langle T_1(g) e_j, e_i \rangle_{V_1}} = 0, \forall u, w \in V_m. \tag{4.6}$$

Therefore, $T_m(x_{i,j}) = 0$.

Also from Proposition 2.2 we obtain that

$$\langle T_1(x_{i,j}) e_k, e_l \rangle_{V_1} = \frac{\dim T_1}{|G|} \sum_{g \in G} \langle T_1(g) e_k, e_l \rangle_{V_1} \overline{\langle T_1(g) e_j, e_i \rangle_{V_1}} = \begin{cases} 1, & \text{if } k = j,\ i = l, \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

Thus, $B(x_{i,j}) = D$ and we have proved that B is surjective.

Exercise 4.3. Show that the map B is injective. Hint: You may consider a regular representation.

□

Exercise 4.4. Describe $B(Z(\mathbb{C}[G]))$.

Exercise 4.5. Suppose that the character of the irreducible representation T_1 is known. Denote by $P \in \mathrm{End}(V)$ a projection operator onto V_1, where

$$V = \bigoplus_{i=1}^{k} V_i \tag{4.8}$$

in Proposition 4.2. Find a formula for $B^{-1}(P)$.

Exercise 4.6. Deduce Proposition 4.1 from Proposition 4.2.

5. External tensor product of representations

Definition 5.1. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(V)$ are representations over the same field. Define an external tensor product $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ of representations T and S by

$$(T \otimes S)(g, h) := T(g) \otimes S(h), \quad \forall g \in G, \forall h \in H. \tag{5.1}$$

It might be confusing that we are using the same symbol to denote the external tensor product and the usual tensor product of representations of the same group.

Proposition 5.2. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(V)$ are representations over the same field. Then the representation $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ is irreducible if and only if both T and S are irreducible.*

Proposition 4.2 implies the following nice corollary.

Proposition 5.3. *Let G be a finite group. Then we have*

$$BiReg \simeq \bigoplus_{T \in \widehat{G}} T \otimes T^*. \tag{5.2}$$

Relevant textbook chapters:

- Fulton, Harris, Sections 2.1 – 2.4;
- James, Liebeck, Chapters 10-16, 18,19;
- Vinberg, Sections 5, 6.