LECTURE 7. GROUP ALGEBRA.

Please treat each unproven statement in this text as an exercise.

1. Algebraic isomorphism of a complex group algebra

In this section, we will prove the fundamental result that we stated earlier in the course.

Proposition 1.1. *Let G be a finite group. The set $\{\chi_T \,|\, T \in \widehat{G}\}$ is an orthonormal basis of $Z(\mathbb{C}[G])$.*

We will deduce it from the following elegant construction.

Proposition 1.2. *Suppose that G is a finite group. Then the set $\widehat{G}$ is finite. Denote $k = |\widehat{G}|$ and $\widehat{G} = \{T_i : G \to \mathrm{GL}(V_i) \,|\, 1 \leq i \leq k\}$. Define*

$$V = \bigoplus_{i=1}^{k} V_i \tag{1.1}$$

Define $B \in \mathrm{Hom}(\mathbb{C}[G], \mathrm{End}(V))$ by

$$B(\sum_{g\in G} a_g g)(v_1, v_2, \ldots, v_k) = (\sum_{g\in G} a_g(T_1(g)(v_1)), \sum_{g\in G} a_g(T_2(g)(v_2)), \ldots, \sum_{g\in G} a_g(T_k(g)(v_k))) \tag{1.2}$$

Then B is an isomorphism between $\mathbb{C}[G]$ and $\prod_{i=1}^{k} \mathrm{End}(V_i)$.

Proof. We need to show that the map B is injective and surjective.

First, let us prove surjectivity. Consider an arbitrary sequence $D = (D_1, \ldots, D_k), \quad D_i \in \mathrm{End}(V_i)$. It suffices to show that we may find a preimage of the sequence D when

$$D_t = 0, \, \forall t \neq s \text{ and } D_s = E_i^j \text{ where } 1 \leq s \leq k, \, 1 \leq i, j \leq \mathrm{dim} T_s. \tag{1.3}$$

Here E_i^j denotes the matrix unit with a 1 in the intersection of the i-th row and the j-th column. Without loss of generality, we may assume that $s = 1$. Denote by $\langle \cdot, \cdot \rangle_{V_1}$ an invariant inner product in the vector space V_1. Let $e_1, \ldots, e_{\mathrm{dim} T_1}$ be the orthonormal basis relative to which we consider E_i^j. The matrix unit E_i^j is determined by the condition

$$\langle E_i^j e_k, e_l \rangle_{V_1} = \begin{cases} 1, & \text{if } k = j, \, i = l, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

Define

$$x_{i,j} := \frac{\mathrm{dim} T_1}{|G|} \sum_{g\in G} \overline{\langle T_1(g) e_j, e_i \rangle_{V_1}} \, g. \tag{1.5}$$

Recall a result from the previous lecture.

Proposition 1.3 (Orthogonality of matrix elements)**.** *Suppose that G is a finite group, $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(W)$ are irreducible complex representations of G, $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ are*

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invariant inner products in the vector spaces V and W respectively, $x, y \in V$, $z, t \in W$. Then

$$\frac{1}{|G|}\sum_{g\in G}\langle T(g)x,y\rangle_V\overline{\langle S(g)z,t\rangle_W} =$$

$$= \begin{cases} 0, & \text{if } T \text{ is not equivalent to } S, \\ \frac{1}{\dim T}\langle x,z\rangle_V\overline{\langle y,t\rangle_V}, & \text{otherwise (here we identify } T \text{ with } S, \text{ and } V \text{ with } W). \end{cases} \tag{1.6}$$

From Proposition 1.3 we get that if $m \neq 1$ then

$$\langle T_m(x_{i,j})u,w\rangle_{V_m} = \frac{\dim T_1}{|G|}\sum_{g\in G}\langle T_m(g)u,w\rangle_{V_m}\overline{\langle T_1(g)e_j,e_i\rangle_{V_1}} = 0, \forall u,w \in V_m. \tag{1.7}$$

Therefore, $T_m(x_{i,j}) = 0$.

Also from Proposition 1.3 we obtain that

$$\langle T_1(x_{i,j})e_k,e_l\rangle_{V_1} = \frac{\dim T_1}{|G|}\sum_{g\in G}\langle T_1(g)e_k,e_l\rangle_{V_1}\overline{\langle T_1(g)e_j,e_i\rangle_{V_1}} = \begin{cases} 1, & \text{if } k=j,\ i=l, \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

Thus, $B(x_{i,j}) = D$, and we have proved that B is surjective.

Exercise 1.4. Show that the map B is injective. Hint: You may consider the regular representation.

□

Exercise 1.5. Describe $B(Z(\mathbb{C}[G]))$.

Exercise 1.6. Suppose that the character of the irreducible representation T_1 is known. Denote by $P \in \mathrm{End}(V)$ a projection operator onto V_1, where

$$V = \bigoplus_{i=1}^{k} V_i \tag{1.9}$$

in Proposition 1.2. Find a formula for $B^{-1}(P)$.

Exercise 1.7. Deduce Proposition 1.1 from Proposition 1.2.

2. External tensor product of representations

Definition 2.1. Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(U)$ are representations over the same field. Define an external tensor product $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ of representations T and S by

$$(T \otimes S)(g,h) := T(g) \otimes S(h), \quad \forall g \in G, \forall h \in H. \tag{2.1}$$

It might be confusing that the same symbol is used to denote both the external tensor product and the usual tensor product of representations of the same group.

Proposition 2.2. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(U)$ are representations over the same field. Then the representation $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ is irreducible if and only if both T and S are irreducible.*

Proposition 1.2 implies the following nice corollary.

Proposition 2.3. *Let G be a finite group. Then we have*

$$BiReg \simeq \bigoplus_{T\in\widehat{G}} T \otimes T^*. \tag{2.2}$$

3. Normal Subgroups and Lifted Representations

Proposition 3.1. *Suppose that H is a normal subgroup of a group G, $p : G \to G/H$ is the canonical projection, and $T : G/H \to \mathrm{GL}(V)$ is a representation of the corresponding quotient group. Define the map*

$$\widetilde{T} : G \to \mathrm{GL}(V) \tag{3.1}$$

by

$$\widetilde{T} := T \circ p. \tag{3.2}$$

Then $\widetilde{T}$ is a representation of the group G. Moreover, $\widetilde{T}$ is irreducible if and only if T is irreducible. The representation $\widetilde{T}$ is called the lift of the representation T.

Exercise 3.2. Express the character of the representation $\widetilde{T}$ using the character of the representation T.

Proposition 3.3. *Suppose that H is a normal subgroup of a group G and that F is a field. Then the map*

$$T \mapsto \widetilde{T} \tag{3.3}$$

is a bijection between the set of representations of the group G/H over the field $\mathbb{F}$ and the set of representations of the group G over the same field satisfying the condition that their kernels contain the subgroup H.

Proposition 3.4. *For every normal subgroup H of a finite group G there exist irreducible complex representations $S_1, \ldots, S_k$ such that*

$$H = \bigcap_{j=1}^{k} \ker S_j.$$

Exercise 3.5. A group is called simple if it has no non-trivial normal subgroups. How can we check if a finite group is simple using its complex character table?

Relevant textbook chapters:

- Fulton, Harris, Sections 2.1 – 2.4;
- James, Liebeck, Chapters 16, 17,19;
- Vinberg, Sections 5, 6.