LECTURE 8. MATRIX LIE GROUPS.

Please treat each unproven statement in this text as an exercise.

1. Exponent of a matrix

Definition 1.1. For $A \in \mathrm{End}(\mathbb{C}^n)$ define its norm by the formula

$$\|A\| := \sqrt{\sum_{1\le i,j\le n} |a_i^j|^2},$$

where (a_i^j) is the matrix of the operator A in the standard basis.

Proposition 1.2. *The normed space* $\mathrm{End}(\mathbb{C}^n)$ *is complete.*

Proposition 1.3. *For any* $A, B \in \mathrm{End}(\mathbb{C}^n)$*, we have the inequalities*

$$\|A+B\| \le \|A\| + \|B\|; \quad \|A\cdot B\| \le \|A\|\cdot\|B\|. \tag{1.1}$$

Proposition 1.4. *For all* $A \in \mathrm{End}(\mathbb{C}^n)$ *the series*

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

converges absolutely in the space $\mathrm{End}(\mathbb{C}^n)$ *with respect to this norm.*

Exercise 1.5. Compute $\exp(A)$ when A is a diagonal matrix.

Exercise 1.6. Compute $\exp(A)$ when A is a Jordan block.

Proposition 1.7. *For any* $A, B \in \mathrm{End}(\mathbb{C}^n)$ *we have the following properties of the exponent:*
a) $\exp(A)$ *is invertible and* $(\exp(A))^{-1} = \exp(-A)$*;*
b) $(\exp(A))^t = \exp(A^t)$*;*
c) if $AB = BA$*, then* $\exp(A+B) = \exp(A)\exp(B)$*;*
d) If B *is invertible, then* $B\exp(A)B^{-1} = \exp(BAB^{-1})$*;*
e) $\exp(sA)$ *is differentiable as a function of* s *and* $\frac{d}{ds}\exp(sA) = A\exp(sA)$*.*

Exercise 1.8. Suppose that V is a complex finite-dimensional vector space and $A \in \mathrm{End(V)}$. Show that $\det(\exp(A)) = e^{\mathrm{tr}(A)}$.

Date: November 8, 2025.

Exercise 1.9. Assume that $A \in \mathrm{End}(\mathbb{C}^n)$, then the map $T_A : \mathbb{R} \to \mathrm{End}(\mathbb{C}^n)$ defined by the formula

$$T_A(s) = \exp(sA)$$

is a representation of the additive group of the field $\mathbb{R}$.

a) Describe conditions on A and n equivalent to the condition that T_A is irreducible.

b) Assume

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

What manifold does the set $\{T_A(s) \,|\, s \in \mathbb{R}\}$ form?

2. Definition and examples

Definition 2.1. Assume that $G \subset GL(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We call G a matrix Lie group if:

1) G is a group with respect to matrix multiplication;

2) G is a closed subset of the topological space $GL(n, \mathbb{F})$.

Consider some examples of matrix Lie groups.

Upper-triangular invertible matrices

$$UT_n(\mathbb{F}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{F}) \mid \det A \neq 0,\ a_i^j = 0,\ \forall 1 \leq i < j \leq n\} \tag{2.1}$$

Unitriangular matrices

$$UUT_n(\mathbb{F}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{F}) \mid \det A \neq 0,\ a_i^j = 0,\ \forall 1 \leq i < j \leq n,\ a_i^i = 1, \forall 1 \leq i \leq n\} \tag{2.2}$$

$UUT_3(\mathbb{C})$ is called the Heisenberg group. General affine group

$$GA_n(\mathbb{F}) = \{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL_n(\mathbb{F}), b \in \mathbb{F}^n\} \tag{2.3}$$

Below are examples of semisimple Lie groups that we will study later. General linear groups over $\mathbb{R}$ and $\mathbb{C}$

$$GL_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \det A \neq 0\} \tag{2.4}$$

$$GL_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid \det A \neq 0\} \tag{2.5}$$

Special linear groups over $\mathbb{R}$ and $\mathbb{C}$

$$SL_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \det A = 1\} \tag{2.6}$$

$$SL_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid \det A = 1\} \tag{2.7}$$

Orthogonal groups over $\mathbb{R}$ and $\mathbb{C}$

$$O_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^t = Id\} \tag{2.8}$$

$$O_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot A^t = Id\} \tag{2.9}$$

Special orthogonal groups over $\mathbb{R}$ and $\mathbb{C}$

$$SO_n(\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^t = Id,\ \det A = 1\} \tag{2.10}$$

$$SO_n(\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot A^t = Id,\ \det A = 1\} \tag{2.11}$$

Symplectic groups over $\mathbb{R}$ and $\mathbb{C}$

$$Sp_n(\mathbb{R}) = \{A \in \mathrm{Mat}(2n \times 2n, \mathbb{R}) \mid A \cdot J \cdot A^t = J\} \tag{2.12}$$

$$Sp_n(\mathbb{C}) = \{A \in \mathrm{Mat}(2n \times 2n, \mathbb{C}) \mid A \cdot J \cdot A^t = J\}, \tag{2.13}$$

where

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}. \tag{2.14}$$

Unitary group and special unitary group

$$U_n = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot \bar{A}^t = Id\} \tag{2.15}$$

$$SU_n = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot \bar{A}^t = Id, \det A = 1\} \tag{2.16}$$

Definition 2.2. A matrix Lie group $G \subset GL(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ is called connected if G is path-connected as a subset of $\mathrm{End}(\mathbb{F}^n)$.

A matrix Lie group $G \subset GL(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ is called compact if G is a compact set as a subset of $\mathrm{End}(\mathbb{F}^n)$.

Exercise 2.3. Which of the Lie groups from the list

$$GL_n(\mathbb{R}), GL_n(\mathbb{C}), SL_n(\mathbb{R}), SL_n(\mathbb{C}), SO_n(\mathbb{R}), SO_n(\mathbb{C}), Sp_n(\mathbb{R}), Sp_n(\mathbb{C}), SU_n$$

are compact?

Exercise 2.4. Show that the Lie group $SO_n(\mathbb{R})$ is connected.

Relevant textbooks chapters:

- Fulton, Harris, Sections 7.1 - 7.3;
- Hall, Sections 1.1 - 1.5, 2.1, 2.2;
- Vinberg, Section 10.