LECTURE 9. LIE ALGEBRAS

Please consider every proposition in this text to be an exercise.

1. The tangent space of a matrix Lie group at the identity

There are two ways to introduce Lie algebras. We will begin with a geometric definition.

Definition 1.1. Suppose that $G \subset \mathrm{GL}(\mathbb{F}^n)$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a matrix Lie group. Consider the set $P_{\mathrm{Id}}G$ (the set of "good" paths) which consists of functions $A : \mathbb{R} \to G \subset \mathrm{Mat}(n \times n, \mathbb{F}) = \mathbb{F}^{n^2}$ such that the following conditions hold:
1) $A(t)$ is smooth as a vector-valued function;
2) $A(0) = \mathrm{Id}$.

Then the tangent space at the point Id is defined as

$$T_{\mathrm{Id}}(G) := \{A'(0) \,|\, A \in P_{\mathrm{Id}}G\}. \tag{1.1}$$

The tangent space $T_{\mathrm{Id}}G$ is called the Lie algebra of the group G and is usually denoted by the corresponding Gothic lowercase letter $\mathfrak{g}$.

Question 1.2. Let $A, B \in P_{\mathrm{Id}}G$. Must the function $A \cdot B$ belong to $P_{\mathrm{Id}}G$?

Exercise 1.3. Let $A, B \in P_{\mathrm{Id}}G$. Express $(A \cdot B)'(t)$ in terms of $A(t), A'(t), B(t)$ and $B'(t)$.

Exercise 1.4. Show that $T_{\mathrm{Id}}G$ is a subspace of the vector space $\mathrm{Mat}(n \times n, \mathbb{F})$.

Thus, the tangent space $T_{\mathrm{Id}}G$ can be considered as a subspace of $\mathbb{R}^{n^2}$, or of $\mathbb{R}^{2n^2}$, if we consider complex matrices. Despite the term "tangent", the space $T_{\mathrm{Id}}G$ does not have to include the element Id at which it is "tangent".

Definition 1.5. Assume $G \subset \mathrm{GL}(\mathbb{F}^n)$ (as everywhere in this section $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a matrix Lie group. Then the dimension of the group G is defined as

$$\dim G := \dim(T_{\mathrm{Id}}G). \tag{1.2}$$

We have not introduced the notions of a manifold or a submanifold. We just note that the dimension of a Lie group G defined above coincides with the dimension of the manifold G.

Exercise 1.6. Find the Lie algebras and their dimensions for the Lie groups $GL_n(\mathbb{F}), GA_n(\mathbb{F})$ (as usual $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).

Usually, the problems of that type can be approached in two steps. Firstly, we deduce that for each "good" path A the derivative $A'(0)$ has to satisfy some conditions. And secondly, for each vector satisfying those conditions, we construct a "good" path with the corresponding derivative. For the second part Proposition 1.7 of the previous lecture might be helpful.

Exercise 1.7. Let $A \in P_{\mathrm{Id}}GL(\mathbb{F}^n)$. Simplify the expression

$$(\det(A(t))')_{t=0}\,. \tag{1.3}$$

Exercise 1.8. Find the Lie algebra and its dimensions for the Lie group $SL_n(\mathbb{F})$ (as usual $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).

Exercise 1.9. Suppose that $G \subset \mathrm{GL}(\mathbb{F}^n)$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a matrix Lie group and $A \in P_{\mathrm{Id}}G$. Show that $(A(t))^{-1}$ is differentiable and express $((A(t))^{-1})'$ in terms of $A'(t)$ and $(A(t))^{-1}$.

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Proposition 1.10. *Assume G is a matrix Lie group and $X, Y \in T_{\mathrm{Id}}G$. Then $XY - YX \in T_{\mathrm{Id}}G$.*

One of the possible approaches to the proof is to consider the function

$$F(t, s) = A(t)B(s)(A(t))^{-1} \tag{1.4}$$

and its partial derivatives. First, differentiate with respect to s, then with respect to t.

Thus, for every matrix Lie group, we have constructed a vector space that is closed with respect to the operation $XY - YX$, which we are going to call its Lie algebra.

2. An algebraic definition of a Lie algebra

Now we will give a more general, algebraic definition of Lie algebra, where the field is arbitrary. We do not claim in this definition that for every Lie algebra, we can construct a Lie group that would correspond to it as in Definition 1.1, or in some other natural sense.

Definition 2.1. Suppose that $\mathfrak{g}$ is a vector space over a field F and that there is a map

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (X, Y) \mapsto [X, Y] \quad (\textit{a commutator})$$

satisfies the following conditions:

(1)

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y],$$
$$\forall X, Y, Z \in \mathfrak{g}, \forall a, b \in F;$$

(2) $[X, X] = 0, \quad \forall X \in \mathfrak{g}$;

(3)

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0,$$
$$\forall X, Y, Z \in \mathfrak{g} \quad (\textit{the Jacobi identity}).$$

Then $\mathfrak{g}$ is called a Lie algebra over the field F. By definition, the dimension of the Lie algebra $\mathfrak{g}$ over the field F is the dimension of the linear space $\mathfrak{g}$ over the field F.

Question 2.2. Is it true that $[Y, X] = -[X, Y]$, $\forall X, Y \in \mathfrak{g}$?

Definition 2.3. Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras over a field $\mathbb{F}$. Then a map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called Lie algebras homomorphism if $\phi \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ and

$$\phi([X, Y]) = [\phi(X), \phi(Y)], \quad \forall X, Y \in \mathfrak{g} \tag{2.1}$$

A bijective homomorphism between Lie algebras is called an isomorphism.

Exercise 2.4. Classify all Lie algebras of dimension 2 up to an isomorphism.

Exercise 2.5. Let A be an associative algebra. Define $[X, Y] := XY - YX$ for arbitrary $X, Y \in A$. Show that A is a Lie algebra.

Exercise 2.6. Suggest a definition of a Lie subalgebra.

Exercise 2.7. Consider the set D of differential operators of the form $\alpha(x)\frac{d}{dx}$, $\alpha(x) \in C^\infty(\mathbb{R})$, where $\alpha(x)\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is defined by

$$((\alpha(x)\frac{d}{dx})(f))(x) = \alpha(x)f'(x)$$

The set D is a Lie algebra with respect to the operation of standard commutator. Show that

$$\mathrm{span}\langle \frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\rangle$$

is a Lie subalgebra of D. Show that this subalgebra is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Sections 8.1, 8.2;
- Hall, Sections 3;
- Vinberg, Section 10.