LECTURE 12. IRREDUCIBLE REPRESENTATIONS OF THE LIE ALGEBRA OF THE LIE GROUP $SL_3(\mathbb{C})$

Please consider every proposition in this text to be an exercise.

1. Weights and roots

Let us recall the structure of an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module.
Define

$$E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \tag{1.1}$$

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{1.2}$$

$$F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

Proposition 1.1. *For every non-negative integer n there exists a unique irreducible $(n+1)$-dimensional representation ρ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In the space V of that representation we can pick a basis $v_0, v_1, \ldots, v_n$ with the following properties:*

$$\rho(H)v_k = (n-2k)v_k, \quad \forall 0 \le k \le n; \tag{1.4}$$

$$\rho(E)v_k = k(n+1-k)v_{k-1}, \quad \forall 1 \le k \le n, \tag{1.5}$$

$$\rho(E)v_0 = 0; \tag{1.6}$$

$$\rho(F)v_k = v_{k+1}, \quad \forall 0 \le k \le n-1, \tag{1.7}$$

$$\rho(F)v_n = 0. \tag{1.8}$$

The number n is called the highest weight of the representation ρ, the vector v_0 is called the highest weight vector, the representation ρ (or, the vector space V with the action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$) is called the highest weight $\mathfrak{sl}_2(\mathbb{C})$-module.

Now we will to study the irreducible $\mathfrak{sl}_3(\mathbb{C})$-module by making a similar construction.

Definition 1.2. Let us denote

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{1.9}$$

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.10}$$

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.11}$$

Exercise 1.3. Find the eigenvectors and eigenvalues of the elements H_1 and H_2 in the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.

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Definition 1.4. Let $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation. Then a pair $\beta = (b_1, b_2)$ is called a weight (of a vector v in the representation π) if $\exists v \in V, v \neq 0$ such that

$$\pi(H_1)(v) = b_1 v, \quad \pi(H_2)(v) = b_2 v. \tag{1.12}$$

Denote by $\alpha_1 = (2, -1)$, $\alpha_2 = (-1, 2)$ and $\alpha_3 = (1, 1)$ the weights of the Lie algebra elements E_1, E_2 and E_3 respectively in the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$. These weights are called roots.

Note that the weights of the elements F_1, F_2 and F_3 in the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$ are equal to $-\alpha_1$, $-\alpha_2$ and $-\alpha_3$ respectively.

Definition 1.5. We say that a weight β is higher than a weight γ if there exist non-negative real numbers p and q such that

$$\beta - \gamma = p\alpha_1 + q\alpha_2. \tag{1.13}$$

Proposition 1.6. *If $v \in V$ is a vector with the weight μ then $\pi(E_i)v$ is also a weight vector with the weight $\mu + \alpha_i$, and $\pi(F_i)v$ is a weight vector with the weight $\mu - \alpha_i$.*

Proposition 1.7. *Assume $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ is a representation. Then there exists a weight vector in V.*

In order to show this proposition, we may use the diagonalizability of $\pi(H_1)$ first. After that we can consider the action of $\pi(H_2)$ on a suitable subspace of highest weight vectors with respect to $\pi(H_1)$.

Actually, we can prove even stronger result.

Proposition 1.8. *Assume $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ is an irreducible representation. Then V can be represented as a direct sum of its weight subspaces.*

Hint: We may consider the direct sum of all weight subspaces and show that it is an invariant subspace of the representation π.

2. Classification Theorem

We need to prove a useful lemma first.

Proposition 2.1. *Assume that $\mathfrak{g}$ is a Lie algebra, $\{x_1, \dots, x_k\}$ is a set of generators of $\mathfrak{g}$, $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation. Then for any sequence $i_1, i_2, \dots, i_m$ of integers from 1 to k we can represent the composition of the endomorphisms as*

$$\pi(x_{i_1}) \dots \pi(x_{i_m}) = \sum_{j_1 + j_2 + \dots + j_k \leq m} a_{j_1, \dots, j_m} (\pi(x_1))^{j_1} \dots (\pi(x_k))^{j_k} \tag{2.1}$$

Proposition 2.2. *Assume $\pi : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(V)$ is an irreducible representation. Then it has a unique highest weight $\mu = (m_1, m_2)$. These numbers m_1, m_2 are non-negative integers. The subspace of vectors with the weight μ is one-dimensional.*

Moreover, every irreducible representation is uniquely determined by its highest weight.

Proposition 2.3. *Two irreducible representations with the same highest weight are isomorphic.*

The following proposition might be useful.

Proposition 2.4. *Assume that $v_1, \ldots, v_k$ are weight vectors with pairwise distinct weights $\beta_1, \ldots, \beta_k$ respectively and*

$$v = \sum_{i=1}^{k} v_i \tag{2.2}$$

is also a weight vector with the weight μ. Then $\exists j$ such that $\mu = \beta_j$ and $v_i = 0$, $\forall 1 \leq i \leq k, i \neq j$.

Now we are going to show that for every pair of non-negative integers (m_1, m_2) there exists a corresponding irreducible representation.

Exercise 2.5. Let $St : \mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^3)$ denote the standard representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Show that it is irreducible and find its highest weight.

Exercise 2.6. Show that the representation $\Lambda^2(St)$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is irreducible and find its highest weight.

Exercise 2.7. Find the highest weight of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.

Proposition 2.8. *For every pair (m_1, m_2) of non-negative integers there exists an irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ with the highest weight (m_1, m_2)*

Textbook chapters relevant to this lecture:

- Fulton, Harris, Sections 8.2, 8.3, 11.1, 11.2, 12, 13;
- Hall, Sections 6.1 - 6.4.
- Vinberg, Section 10, 11.