# LECTURE 13. IRREDUCIBLE COMPLEX CHARACTERS OF SYMMETRIC GROUPS

## 1. Partitions

**Definition 1.1.** We will call a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ a partition of $n$ if:

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$;
(2) $|\lambda| := \sum_{j=1}^{k} \lambda_j = n$.

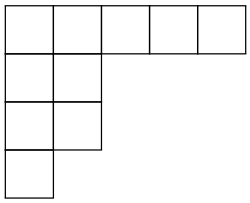
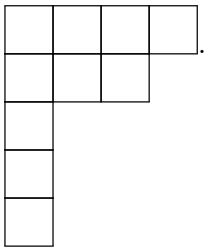
The length of a partition is defined as $l(\lambda) := k$. Denote

$$m_j(\lambda) := \#\{i \,|\, \lambda_i = j\} \tag{1.1}$$

Also we are going to write that $|\lambda| = n$.

With every partition $\lambda$ we associate a Young diagram $D_\lambda$ with $\lambda_j$ boxes in the $j$-th row, the rows of boxes lined up on the left. The conjugate partition $\lambda'$ is obtained by interchanging the rows and the columns in the Young diagram $D_\lambda$.

For example, consider $\lambda = (5, 2, 2, 1)$. Then $\lambda' = (4, 3, 1, 1, 1)$, and the corresponding Young diagrams look as follows:

$D_\lambda =$  , $D_{\lambda'} =$  . (1.2)

Denote by $p(n)$ the number of partitions of the number $n$.

**Definition 1.2.** Let $\rho$ be a partition of $n$. Denote by $M_\rho$ the subset of the group $S_n$ consisting of elements of the cycle type $\rho$.

**Proposition 1.3.** *For a partition $\rho$ of $n$, denote*

$$z_\rho = \prod_{j=1}^{l(\rho)} j^{m_j(\rho)}((m_j(\rho))!) \tag{1.3}$$

*Show that*

$$|M_\rho| = \frac{n!}{z_\rho}. \tag{1.4}$$

## 2. Symmetric Functions

Consider two important families of symmetric polynomials.

**Definition 2.1.** Define Newton symmetric polynomials as

$$p_k(x_1, x_2, \dots, x_m) := \sum_{j=1}^{m} x_j^k; \quad p_\rho(x_1, x_2, \dots, x_m) := \prod_{i=1}^{l(\rho)} p_{\rho_i}. \tag{2.1}$$

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**Definition 2.2.** If $l(\lambda) \leq m$ we define Schur symmetric polynomials as

$$s_\lambda(x_1, x_2, \ldots, x_m) := \frac{\det(x_j^{\lambda_i+m-i})_{1\leq i,j\leq m}}{\det(x_j^{m-i})_{1\leq i,j\leq m}} = \frac{\det(x_j^{\lambda_i+m-i})_{1\leq i,j\leq m}}{\prod_{1\leq i<j\leq m}(x_i - x_j)}. \tag{2.2}$$

## 3. Frobenius Formula for Irreducible Complex Characters of Symmetric Groups

There is a natural way to parametrize irreducible complex representations of the symmetric group by the partitions of the number $n$. Let us briefly discuss it.

**Proposition 3.1.** *Denote by $\eta_n$ the trivial character of $S_n$. We will assume that $\eta_k = 0$ if $k < 0$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $n$ and define*

$$S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}. \tag{3.1}$$

*Then*

$$\eta_{\lambda_1} \cdot \eta_{\lambda_2} \ldots \eta_{\lambda_k} \tag{3.2}$$

*is a trivial character of $S_\lambda$. Denote by $\eta_\lambda$ the character of $S_n$ induced by the trivial character of $S_\lambda$. Define*

$$\chi^\lambda := \det[\eta_{\lambda_i - i+j}]_{1\leq i,j\leq k}, \tag{3.3}$$

*where each summand in the determinant expansion stands for the character that we will induce to the group $S_n$. Then $\chi^\lambda$ is an irreducible complex character of the group $S_n$, and we obtain a bijection between the set of partitions of the number $n$ and the set $\widehat{S_n}$.*

**Definition 3.2.** Suppose $\lambda$ and $\rho$ are partitions of $n$. Let us introduce the following notation:

$$\alpha^\lambda := \frac{1}{|S_n|}\sum_{g\in S_n} \chi^\lambda(g) g \in Z(\mathbb{C}[S_n]); \quad C_\rho := \frac{1}{|M_\rho|}\sum_{g\in M_\rho} g \in Z(\mathbb{C}[S_n]). \tag{3.4}$$

Also let us denote by $\chi^\lambda_\rho$ the value of the character $\chi^\lambda$ at the element of the cycle type $\rho$.

**Proposition 3.3.** *Let $\lambda$ be a partition of $n$. Then we have the following equality*

$$\alpha^\lambda = \sum_{|\rho|=n} \chi^\lambda_\rho \frac{C_\rho}{z_\rho}. \tag{3.5}$$

**Proposition 3.4.** *Let $\lambda$ be a partition of a number $n$. Then we have the equality*

$$s_\lambda = \sum_{|\rho|=n} \chi^\lambda_\rho \frac{p_\rho}{z_\rho}. \tag{3.6}$$

*Let $\rho$ be a partition of $n$. Then we get*

$$p_\rho = \sum_{|\lambda|=n} \chi^\lambda_\rho s_\lambda. \tag{3.7}$$

## 4. Dimension of a Partition

**Definition 4.1.** Consider a partition $\lambda$ of a number $n$. A standard tableau of the shape $\lambda$ is a way to fill in all the boxes of its Young diagram $D_\lambda$ with the numbers $1, \dots, n$ so that the following conditions hold:

(1) each number is written only in one box;
(2) the numbers go in a strictly increasing order along each row;
(3) the numbers go in a strictly increasing order along each column.

**Definition 4.2.** Consider a partition $\lambda$ of a number $n$. The number of standard tableaux of the shape $\lambda$ is called the dimension of the partition $\lambda$ and is denoted by $\dim\lambda$.

Here is an example of the standard tableau of the shape $(5, 2, 2, 1)$:

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 5 | 6 | 9 |
| 2 | 7 | | | |
| 4 | 10 | | | |
| 8 | | | | |

**Proposition 4.3.** *Let $|\mu| = n$. Then we have*

$$s_\mu \cdot p_{(1)} = \sum_\lambda s_\lambda, \tag{4.1}$$

*where the sum runs over the partitions $\lambda$ such that $|\lambda| = n + 1$ and $D_\mu \subset D_\lambda$.*

It implies the following identity for the dimensions of partitions.

**Proposition 4.4.**

$$p_{(1)}^n = \sum_{\lambda \vdash n} \dim\lambda \cdot s_\lambda. \tag{4.2}$$

That result allows us to obtain a closed formula for the dimension of a partition.

**Proposition 4.5.** *Let $\lambda$ be a partition. Then*

$$\dim\lambda = \frac{|\lambda|!}{\prod_{j=1}^{l(\lambda)}(\lambda_j + l(\lambda) - j)!} \prod_{1 \le i < j \le l(\lambda)} (\lambda_i - \lambda_j + j - i). \tag{4.3}$$

**Definition 4.6.** For a cell $(i, j) \in D_\lambda$ define its hook length as

$$h(i, j) := \lambda_i + \lambda'_j - i - j + 1. \tag{4.4}$$

**Proposition 4.7.** *(Hook length formula) Let $\lambda$ be a partition. Then*

$$\dim\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in D_\lambda} h(i, j)}. \tag{4.5}$$

**Definition 4.8.** Consider a partition $\lambda$ of a number $n$ and an arbitrary positive integer $k$. Let us fill in each box of the Young diagram $D_\lambda$ with one of the numbers $1, \dots, k$, The result is called a semistandard tableau of shape $\lambda$ (filled with the numbers $1, \dots, k$) if the entries weakly increase along each row and strictly increase down each column. For a semistandard tableau $T$ and some variables $x_1, \dots, x_k$ put

$$x^T := \prod_{j=1}^{k} x_j^{m_j(T)}, \tag{4.6}$$

where $m_j(T)$ is the number of entries in the table $T$ that are equal to $j$.

We can obtain the following combinatorial description of Schur functions.

**Proposition 4.9.** *Suppose that $\lambda$ is a partition, $k \geq l(\lambda)$, $x_1, \ldots, x_k$ are some variables. Then*

$$s_\lambda(x_1, \ldots, x_k) = \sum_T x^T, \tag{4.7}$$

*where the sum is taken over semistandard tableau of shape $\lambda$ filled with the numbers $1, \ldots, k$.*

## 5. Murnaghan-Nakayama Rule

We can obtain a combinatorial algorithm how to compute $\chi^\lambda_\rho$.

**Definition 5.1.** Let $D_\mu \subset D_\lambda$. Then the pair $(\mu, \lambda)$ (the difference $D_\lambda \setminus D_\mu$) is called a border strip if the following conditions hold:

(1) $D_\lambda \setminus D_\mu$ has no $2 \times 2$ squares;
(2) $D_\lambda \setminus D_\mu$ is connected, where we assume that the path can go either horizontally or vertically.

Denote by $\mathrm{height}(\lambda \setminus \mu)$ the number of rows $D_\lambda \setminus D_\mu$ touches minus one.

**Proposition 5.2.** *Let $|\mu| = n$. Then we get*

$$s_\mu \cdot p_r = \sum_\lambda (-1)^{\mathrm{height}(\lambda \setminus \mu)} s_\lambda, \tag{5.1}$$

*where the sum runs over the partitions $\lambda$ such that $|\lambda| = n + r$, $D_\mu \subset D_\lambda$ and $D_\lambda \setminus D_\mu$ is a border strip .*

**Proposition 5.3.** *(Murnaghan–Nakayama rule) Let $|\lambda| = |\rho| = n$. Then we get*

$$\chi^\lambda_\rho = \sum_T (-1)^{\mathrm{height(T)}}, \tag{5.2}$$

*where the sum runs over the sequences of partitions*

$$T = (\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \ldots \lambda^{(l(\rho)-1} \subset \lambda^{(l(\rho)} = \lambda) \tag{5.3}$$

*such that*

(1) *$(\lambda^{(j-1)}, \lambda^{(j)})$ is a border strip for $1 \leq j \leq l(\rho)$;*
(2) *$|\lambda^{(j)}| = |\lambda^{(j-1)}| + \rho_j$ for $1 \leq j \leq l(\rho)$;*

*and*

$$\mathrm{height(T)} := \sum_{\mathrm{j}=1}^{\mathrm{l}(\rho)} \mathrm{height}(\lambda^{(\mathrm{j})} \setminus \lambda^{(\mathrm{j}-1)}). \tag{5.4}$$