PROBLEM SET 1, DUE SEPTEMBER 20

1. The Notion of a Representation

Problem 1.1. *Let V be a vector space over a field $\mathbb{F}$. Show that the set of invertible linear maps from V to itself is a group with respect to the operation of map composition.*

Problem 1.2. *Suppose that P is a linear operator in a vector space V. Show that $P^2 = P$ if and only if the following two conditions hold:*

(1) $V = \ker P \oplus \mathrm{im} P$;
(2) $Px = x, \quad \forall x \in \mathrm{im} P$.

P is called a projection operator.

Problem 1.3. *Find all 1-dimensional representations of the cyclic group $\mathbb{Z}_n$:*
a) over $\mathbb{R}$, b) over $\mathbb{C}$.

Problem 1.4. *Consider the representation $T : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}^2)$ given by the formula*

$$T(k) = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\ -\sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Find all its non-trivial invariant subspaces.

Problem 1.5. *Suppose that $T : G \to \mathrm{GL}(V)$ is a representation and $U \subset V$ is an invariant subspace. Provide a definition of the corresponding quotient representation in the space V/U and show that it is well defined.*

Problem 1.6. *Which of the following representations are pairwise isomorphic:*

a) $T(n) = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}, \quad a \neq 0, b \neq 0,$

b) $T(n) = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix},$

c) $T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}, \quad a,\ b,\ \alpha \in \mathbb{R}?$

Problem 1.7. *Does there exist an irreducible real representation T of the group $\mathbb{Z}_2$ such that $\dim T \geq 2$?*

Problem 1.8. *Does there exist a real representation T of the group $\mathbb{Z}_3 = \{0, 1, 2\}$ such that*

$$T(1) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}?$$

Problem 1.9. *Does there exist a representation of a finite group that is not completely reducible?*

Problem 1.10. *Define a representation T of the symmetric (permutation) group S_n in $\mathbb{C}^n$ by*

$$T(\sigma)((x_1, \ldots, x_n)) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

for $\sigma \in S_n$. Find all non-trivial invariant subspaces of the representation T.

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