PROBLEM SET 2, DUE SEPTEMBER 27

2. Schur's Lemma. Maschke's Theorem.

Problem 2.1. *Describe all morphisms between any two one-dimensional complex representations of the group* $\mathbb{Z}_n$.

Problem 2.2. *Suppose that* $T : G \to \mathrm{GL}(V)$ *and* $S : G \to \mathrm{GL}(W)$ *are representations,* $A \in \mathrm{Mor}(T, S)$ *and* $U \subset V$ *is an invariant subspace of the representation* T. *Show that*

$$AU := \{Au \mid u \in U\}$$

is an invariant subspace of the representation S.

Problem 2.3. *Suppose that* $T : G \to \mathrm{GL}(V)$ *and* $S : G \to \mathrm{GL}(W)$ *are representations,* $A \in \mathrm{Mor}(T, S)$ *and* $U \subset W$ *is an invariant subspace of the representation* S. *Does the set*

$$A^{-1}U := \{x \in V \mid Ax \in U\}$$

have to be an invariant subspace of the representation T*?*

Problem 2.4. *Suppose that* $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{C}^2)$ *and* $S : \mathbb{Z} \to \mathrm{GL}(\mathbb{C}^2)$ *are defined as*

$$T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}, \quad S(n) = \begin{pmatrix} \exp(in\alpha) & 0 \\ 0) & \exp(-in\alpha) \end{pmatrix}.$$

Describe $\mathrm{Mor}(T, S)$.

Problem 2.5. *Suppose that* $T : \mathbb{Z}_4 \to \mathrm{GL}(\mathbb{C}^3)$ *and* $S : \mathbb{Z}_4 \to \mathrm{GL}(\mathbb{C}^3)$ *are defined as*

$$T(n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & i^n \end{pmatrix}, \quad S(n) = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & i^n & 0 \\ 0 & 0 & i^n \end{pmatrix}.$$

Describe $\mathrm{Mor}(T, S)$ *and* $\mathrm{End}(S)$.

Problem 2.6. *Suppose that* $T : \mathbb{Z}_4 \to \mathrm{GL}(\mathbb{R}^4)$ *is defined as*

$$T(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Describe $\mathrm{End}(S)$.

Problem 2.7. *Is it true that every irreducible complex representation of an abelian group is one-dimensional?*

Problem 2.8. *Find all one-dimensional complex representations of the group* $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Problem 2.9. *Describe all finite-dimensional division algebras over* $\mathbb{C}$.

Problem 2.10 (Challenging)**.** *Describe all finite-dimensional division algebras over* $\mathbb{R}$.

Problem 2.11 (Challenging)**.** *Construct an irreducible real representation of a finite group such that its algebra of endomorphisms is isomorphic to the division algebra of quaternions. You may pick any finite group you prefer.*

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Problem 2.12. *Show that*

$$U \oplus \ker P_0 = V \tag{2.1}$$

in the proof of Maschke's Theorem.

Problem 2.13. *Suppose that the group G is finite and $T : G \to \mathrm{GL}(\mathbb{C}^2)$ is a representation. Suppose $\exists g, h \in G$ such that $T(g)T(h) \neq T(h)T(g)$. Can the representation T be not irreducible?*