PROBLEM SET 3, DUE OCTOBER 4

3. Decomposition of a Representation; One-dimensional Representations.

Problem 3.1. *Suppose that T is a complex representation of the group G and*

$$T \cong \bigoplus_{S \in \widehat{G}} m_s S. \tag{3.1}$$

Then

$$m_s = \dim(\mathrm{Mor}(S,T)) = \dim(\mathrm{Mor}(T,S)), \quad \forall S \in \widehat{G}. \tag{3.2}$$

Problem 3.2. *The quaternion group Q_8 is generated by the elements i, j, ε such that $\varepsilon^2 = e$, $\varepsilon i = i\varepsilon$, $\varepsilon j = j\varepsilon$, $i^2 = j^2 = \varepsilon$, $ij = \varepsilon ji$. Find all its 1-dimensional complex representations.*

Problem 3.3. *The dihedral group D_n is generated by a rotation a and a reflection s such that $a^n = s^2 = e$, $sas = a^{-1}$. Find all 1-dimensional complex representations of the group D_n. Note that the answer depends on the parity of n.*

Problem 3.4. *Prove that $[S_n, S_n] = A_n$, where $A_n := sgn^{-1}(1)$.*

Problem 3.5. *Find all 1-dimensional complex representations of the group S_n.*

Problem 3.6. *Find all 1-dimensional complex representations of the group A_4.*

Problem 3.7. *Let $\{T_1, \dots, T_n\}$ be the set of all pairwise non-isomorphic one-dimensional complex representations of G. Prove that this set forms a group with respect to the elementwise multiplication. Describe this group for $G = \mathbb{Z}_n$, S_n.*

Problem 3.8. *Show that*

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \cdot \mathrm{tr}(B). \tag{3.3}$$

Is there an analogous formula for the determinants?

Problem 3.9. *Let $\dim V = n$. Find a formula for $\dim S^k(V)$.*

Problem 3.10. *Show that T^* is indeed a representation of the group G.*

Problem 3.11. *Define $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^3)$ as*

$$T(n) = \begin{pmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that T is a representation. Find the matrix of $T^*(n)$ in the dual basis.

Problem 3.12. *Let $T : G \to \mathrm{GL}(V)$ be a representation. Is it always true that the representations T and $(T^*)^*$ are isomorphic?*

Problem 3.13. *Let T be an irreducible representation of a group G. Is it possible that the representation T^* is not irreducible?*

Problem 3.14. *Let T be a representation of a finite group G. Is it true that the representation $T \otimes T^*$ of the group G always has a one-dimensional invariant subspace?*

Problem 3.15. *Suppose that T is a representation of a group G over a field $\mathbb{F}$ and $\mathrm{char}\mathbb{F} \neq 2$. Show that*

$$T^{\otimes 2} = \Lambda^2 T \oplus S^2 T. \tag{3.4}$$

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