PROBLEM SET 4, DUE OCTOBER 11

4. Complexification

Problem 4.1. *Suppose that $T : G \to \mathrm{GL}(V)$ is a representation, and $U, W \subset V$ are invariant subspaces. Are $U \cap W$ and $U + W$ also always invariant subspaces?*

Problem 4.2. *Let V be a real vector space. Show that $\forall v \in V^{\mathbb{C}}$ $\exists u, w \in V$ such that*

$$v = 1 \otimes u + i \otimes w.$$

Problem 4.3. *Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V < \infty$, then the dimension of $V^{\mathbb{C}}$ over $\mathbb{C}$ equals $\dim V$. What can be said about the set $V_0^{\mathbb{C}}$ and its dimension?*

Problem 4.4. *Let V be a real vector space. Show that*

$$\mathrm{Conj} \circ A^{\mathbb{C}} = A^{\mathbb{C}} \circ \mathrm{Conj}, \quad \forall A \in \mathrm{End}(V).$$

Problem 4.5. *Suppose that $T : G \to \mathrm{GL}(V)$ is a real representation and $W \subset V^{\mathbb{C}}$ is an invariant subspace of the representation $T^{\mathbb{C}}$. Is the set $\mathrm{Conj}(W)$ always an invariant subspace of the representation $T^{\mathbb{C}}$?*

Problem 4.6. *Let $T : G \to \mathrm{GL}(V)$ be an irreducible real representation of odd dimension. Then its complexification $T^{\mathbb{C}}$ is also irreducible.*

Problem 4.7. *Consider an equilateral triangle on the plane with its center at the origin. The symmetric group S_3 acts on this triangle by permutations of its vertices. Show that the corresponding 2-dimensional representation T of the group S_3 over $\mathbb{R}$ is irreducible. Also show that the complexification $T^{\mathbb{C}}$ is irreducible as well.*

Problem 4.8. *Consider a regular tetrahedron in the Euclidean space $\mathbb{R}^3$ with the center at the origin. Denote by $O(T)$ the set of isometries of $\mathbb{R}^3$ that preserve the tetrahedron. Let*

$$SO(T) = \{A \in O(T) \mid \det(A) > 0\}.$$

Show that

$$O(T) \simeq S_4, \quad SO(T) \simeq A_4.$$

Problem 4.9. *Consider a cube in the Euclidean space $\mathbb{R}^3$ with the center at the origin. Denote by $O(C)$ the set of isometries of $\mathbb{R}^3$ that preserve the cube. Define*

$$SO(C) = \{A \in O(C) \mid \det(A) > 0\}.$$

Show that

$$O(C) \simeq S_4 \times \mathbb{Z}_2, \quad SO(C) \simeq S_4.$$

Problem 4.10. *Show that the three-dimensional real representations of the group A_4 that are determined by the motions of the regular tetrahedron and the cube are irreducible.*

Problem 4.11. *Is the representation of the group S_4 in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?*

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Problem 4.12. *Is the representation of the group A_4 in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?*

Problem 4.13. *Let $k \in \mathbb{Z}$. Define a 2-dimensional complex representation T_k of the group D_n as*

$$T_k(a) = \begin{pmatrix} \cos\frac{2\pi k}{n} & \sin\frac{2\pi k}{n} \\ -\sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{pmatrix}, \qquad T_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Check that the representation T_k is well-defined for all integers k. Determine when the representation T_k is irreducible.