PROBLEM SET 5, DUE OCTOBER 18

5. Character of a Representation

Problem 5.1. *Let $T : G \to \mathrm{GL}(V)$ be a complex representation of a finite group G. Show that*

$$|\chi_T(g)| \leq \dim T, \quad \forall g \in G.$$

Problem 5.2. *Suppose that G is a finite group and that $x, y \in G$. Show that*

$$\sum_{T \in \widehat{G}} \chi_T(x)\overline{\chi_T(y)} = \begin{cases} \frac{|G|}{|C|}, & \text{if } x \text{ and } y \text{ belong to the same conjugacy class } C, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 5.3. *Construct the complex character table of the group A_4.*

Problem 5.4. *Construct the complex character table of the group S_4.*

Problem 5.5. *Describe the conjugacy classes of the group D_n.*

Problem 5.6. *Construct the complex character table of the group D_n when n is odd.*

Problem 5.7. *Construct the complex character table of the group D_n when n is even.*

Problem 5.8. *Find the multiplicities of the decomposition of every tensor product of two irreducible complex representations (they may be equal) of the group S_4 into the sum of irreducible ones.*

Problem 5.9. *For every irreducible complex representation T of the group S_4 find the multiplicities of the decomposition of the representations S^2T and $\Lambda^2 T$ into the sum of irreducible ones.*

Problem 5.10. *Consider a set F of complex functions on the set M of the faces of the cube. It is clear that F is a vector space with respect to the addition and the multiplication by complex numbers defined by the following formulas:*

$$(\lambda \cdot f)(x) := \lambda(f(x)), \qquad (f+g)(x) := f(x) + g(x), \qquad \forall f, g \in F, \forall x \in M, \forall \lambda \in \mathbb{C}.$$

Recall that $S_4 \cong SO(C)$, which is the group of orientation-preserving isometries of the cube. For arbitrary elements $g \in S_4$ and $x \in M$ denote by gx the image of the face x after applying the rotation from $SO(C)$ corresponding to g. Denote by L a representation of the group S_4 in the vector space F given by the following formula

$$(L(g)f)(x) := f(g^{-1}x), \qquad \forall f \in F, \forall x \in M, \forall g \in S_4.$$

Find the multiplicities of the decomposition of the representation L into the sum of irreducible ones.

Problem 5.11. *Describe minimal invariant subspaces of the representation L from the previous problem.*

Problem 5.12. *Suppose V and W are complex inner product vector spaces, $f : V \times W \to \mathbb{C}$ is a sesquilinear function, i.e. f is linear with respect to the first variable and conjugate linear with respect to the second variable. Then $\exists A \in \mathrm{Hom}(V, W)$ such that*

$$f(x, z) = \langle Ax, z \rangle_W.$$

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