PROBLEM SET 6, DUE OCTOBER 25

6. Orthogonality Relations

Problem 6.1. *As a vector space, the group algebra is isomorphic to the space of functions on the group introduced in Lecture 5. How can we interpret multiplication in the group algebra in terms of functions?*

Problem 6.2. *Give a definition of a representation of an algebra. Also give definitions of invariant subspaces of such representations. For a representation $T : G \to \mathrm{GL}(V)$ of a group G over a field $\mathbb{F}$, show that T is irreducible if and only if the corresponding representation $\overline{T}$ of its group algebra $\mathbb{F}[G]$ is irreducible.*

Problem 6.3. *Is there an irreducible representation T of a group G over a field $\mathbb{F}$ such that*

$$\overline{T}(\mathbb{F}[G]) \neq \mathrm{End}(V)?$$

Problem 6.4. *Find the character of the representation* BiReg.

Problem 6.5. *Construct the complex character table of the group Q_8.*

Problem 6.6. *Let T be an irreducible real representation of an abelian group G. Is the inequality $\dim T \leq 2$ always true?*

Problem 6.7. *Do there exist two non-isomorphic groups G and H such that $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are isomorphic as algebras?*

Problem 6.8. *Let n be even. Find the multiplicities of the decomposition of every tensor product of two irreducible complex representations (they may be equal) of the group D_n into the sum of irreducible ones.*

Problem 6.9. *Consider the group*

$$G = \langle a, b \,|\, a^5 = b^4 = e, bab^{-1} = a^2 \rangle.$$

Find its complex character table.

Problem 6.10. *Does there exist a group G such that $|G| = 24$ and $[G, G] = G$?*

Problem 6.11. *Let G be a finite group. Consider the left regular representation*

$$LReg : G \to \mathrm{GL}(\mathbb{C}[G]). \tag{6.1}$$

Describe the minimal invariant subspace of $\mathbb{C}[G]$ corresponding to the trivial subrepresentation.

Problem 6.12. *A representation T is called faithful if $\ker(T) = \{e\}$. How do we determine whether a complex representation of a finite group is faithful using the values of its character?*

Problem 6.13. *Consider the set F of complex functions on the set M of the vertices of the cube. It is clear that F is a vector space with respect to the addition and the multiplication by complex numbers defined by the following formulas:*

$$(\lambda \cdot f)(x) := \lambda(f(x)), \qquad (f + g)(x) := f(x) + g(x), \qquad \forall f, g \in F, \forall x \in M, \forall \lambda \in \mathbb{C}.$$

Recall that $S_4 \cong SO(C)$, which is the group of orientation-preserving isometries of the cube. For arbitrary elements $g \in S_4$ and $x \in M$ denote by gx the image of the vertex x after applying the

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rotation from $SO(C)$ corresponding to g. Denote by L a representation of the group S_4 in the vector space F given by the following formula

$$(L(g)f)(x) := f(g^{-1}x), \qquad \forall f \in F, \forall x \in M, \forall g \in S_4.$$

Find the multiplicities of the decomposition of the representation L into the sum of irreducible ones. Describe minimal invariant subspaces of this representation.

Problem 6.14. *Let us write six numbers $a_1, a_2, \ldots, a_6$ on the faces of a cube. On the first step, replace the number on each face with the average of the numbers on the neighboring faces. On the second step, repeat this procedure using the numbers obtained after the first step, and continue similarly for subsequent steps. What is the easiest way to find the numbers written on the cube after n steps?*