PROBLEM SET 7, DUE NOVEMBER 8

7. Group Algebra

Problem 7.1. *Deduce Proposition 1.1 of Lecture 7 from Proposition 1.2.*

Problem 7.2. *How many elements $x \in Z(CC[S_4])$ satisfy $x^2 = x$?*

Problem 7.3. *Suppose that every irreducible complex representation of a finite group G is one-dimensional. Does G have to be abelian?*

Problem 7.4. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : H \to \mathrm{GL}(U)$ are representations over the same field. Show that the representation $T \otimes S : G \times H \to \mathrm{GL}(V \otimes U)$ is irreducible if and only if both T and S are irreducible.*

Problem 7.5. *Let G be a finite group. Show that*

$$BiReg \simeq \bigoplus_{T \in \widehat{G}} T \otimes T^*. \tag{7.1}$$

Problem 7.6. *Consider the group*

$$T_{4n} = \langle a, b \,|\, a^{2n} = e, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

Describe all 1-dimensional complex representations of this group.

Problem 7.7. *Suppose that $T : G \to \mathrm{GL}(V)$ is a complex representation of a finite group G and $\chi_T(g)$ is an even integer for every $g \in G$. Must there exist a representation $S : G \to \mathrm{GL}(U)$ such that*

$$T \simeq 2S\,? \tag{7.2}$$

Problem 7.8. *Suppose $T : G \to \mathrm{GL}(V)$ is a complex representation of a finite group G and*

$$\det(T(h)) = -1 \tag{7.3}$$

for some element $h \in G$. Show that there exists a subgroup of index 2 in G.

Problem 7.9. *Let g be an element of order 2 in the finite group G and let χ be the character of a complex representation of G. Show that $\chi(g)$ is an integer and exactly one of the two conditions holds: either*

$$\chi(g) \equiv \chi(e) (mod 4), \tag{7.4}$$

or G has a normal subgroup of index 2.

Problem 7.10. *Show that for every normal subgroup H of a finite group G there exist irreducible complex characters $\chi_1, \ldots, \chi_k$ such that*

$$H = \bigcap_{j=1}^{k} \{g \in G \mid \chi_j(g) = \chi_j(e)\}$$

Problem 7.11. *A group is called simple if it has no non-trivial normal subgroups. How can one check if a finite group is simple using its complex character table?*

Problem 7.12. *Suppose that T is a faithful complex representation of a finite group G. Then for every irreducible complex representation $S \in \widehat{G}$ there exists a non-negative integer k such that S appears in the decomposition of $T^{\otimes k}$ with a non-zero multiplicity.*

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