PROBLEM SET 8, DUE NOVEMBER 15

## 8. Matrix Lie Groups

**Problem 8.1.** *Show that the normed space* $\mathrm{End}(\mathbb{C}^n)$ *is complete.*

**Problem 8.2.** *For any* $A, B \in \mathrm{End}(\mathbb{C}^n)$*, we have the following inequalities:*

$$\|A+B\| \leq \|A\| + \|B\|; \quad \|A \cdot B\| \leq \|A\| \cdot \|B\|. \tag{8.1}$$

**Problem 8.3.** *Show that for all* $A \in \mathrm{End}(\mathbb{C}^n)$*, the series*

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

*converges absolutely in the space* $\mathrm{End}(\mathbb{C}^n)$ *with respect to this norm.*

**Problem 8.4.** *Compute* $\exp(A)$ *when* $A$ *is a complex matrix in its Jordan canonical form.*

**Problem 8.5.** *Suppose that* $V$ *is a complex finite-dimensional vector space and* $A \in \mathrm{End(V)}$*. Show that* $\det(\exp(A)) = e^{\mathrm{tr}(A)}$*.*

**Problem 8.6.** *Assume that* $A \in \mathrm{End}(\mathbb{C}^n)$*, then the map* $T_A : \mathbb{R} \to \mathrm{End}(\mathbb{C}^n)$ *defined by the formula*

$$T_A(s) = \exp(sA)$$

*is a representation of the additive group of the field* $\mathbb{R}$*.*

*a) Describe conditions on* $A$ *and* $n$ *that are equivalent to the condition that* $T_A$ *is irreducible.*

*b) Assume*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

*What manifold is formed by the set* $\{T_A(s) \,|\, s \in \mathbb{R}\}$*?*

**Problem 8.7.** *Show that the groups introduced in Section 2 of Lecture 8 are indeed matrix Lie groups.*

**Problem 8.8.** *Which of the Lie groups from the list*

$$GL_n(\mathbb{R}), GL_n(\mathbb{C}), SL_n(\mathbb{R}), SL_n(\mathbb{C}), SO_n(\mathbb{R}), SO_n(\mathbb{C}), Sp_n(\mathbb{R}), Sp_n(\mathbb{C}), SU_n,$$

*are compact?*

**Problem 8.9.** *Which of the Lie groups* $UT_n(\mathbb{R}), UT_n(\mathbb{C}), UUT_n(\mathbb{R}), UUT_n(\mathbb{C})$ *are connected?*

**Problem 8.10.** *Which of the Lie groups* $GL_n(\mathbb{R}), GL_n(\mathbb{C}), SL_n(\mathbb{R}), SL_n(\mathbb{C})$ *are connected?*

**Problem 8.11.** *Which of the Lie groups* $O_n(\mathbb{R}), U_n, SO_n(\mathbb{R}), SU_n$ *are connected?*

**Problem 8.12.** *Which well-known sets are the groups* $U_1, O_2(\mathbb{R}), SO_2(\mathbb{R})$*, and* $SU_2$*?*

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