PROBLEM SET 9, DUE NOVEMBER 22

9. Lie Algebras

Problem 9.1. *Let G be a matrix Lie group. Show that its connected component of the identity is a normal subgroup of G.*

Problem 9.2. *Find the Lie algebra and its dimension for the Lie group $GA_n(\mathbb{R})$.*

Problem 9.3. *Let $A \in P_{\mathrm{Id}}GL(\mathbb{F}^n)$. Simplify the expression*

$$(\det(A(t))')_{t=0}\,. \tag{9.1}$$

Problem 9.4. *Find the Lie algebras and their dimensions for the Lie group $SL_n(\mathbb{F})$ (as usual $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).*

Problem 9.5. *Suppose that $G \subset \mathrm{GL}(\mathbb{F}^n)$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a matrix Lie group and $A \in P_{\mathrm{Id}}G$. Show that $(A(t))^{-1}$ is differentiable and express $((A(t))^{-1})'$ in terms of $A'(t)$ and $(A(t))^{-1}$.*

Problem 9.6. *Suppose that G is a matrix Lie group and $X, Y \in T_{\mathrm{Id}}G$. Show that $XY - YX \in T_{\mathrm{Id}}G$.*

Problem 9.7. *Describe the Lie algebra of the Lie group of the invertible diagonal matrices.*

Problem 9.8. *Describe the Lie algebras and find their dimensions for the Lie groups $UT_n(\mathbb{F}), UUT_n(\mathbb{F})$.*

Problem 9.9. *Find the Lie algebras and their dimensions for the Lie groups $SO_n(\mathbb{R})$ and $SO_n(\mathbb{C})$.*

Problem 9.10. *Find the Lie algebra and its dimensions of the Lie group SU_n.*

Problem 9.11. *Classify all Lie algebras of dimension 2 up to isomorphism.*

Problem 9.12. *Let A be an associative algebra. Define $[X, Y] := XY - YX$ for arbitrary $X, Y \in A$. Show that A is a Lie algebra.*

Problem 9.13. *Suggest a definition of a Lie subalgebra.*

Problem 9.14. *Consider the set D of differential operators of the form $\alpha(x)\frac{d}{dx}$, $\alpha(x) \in C^\infty(\mathbb{R})$, where $\alpha(x)\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is defined by*

$$((\alpha(x)\frac{d}{dx})(f))(x) = \alpha(x)f'(x)$$

The set D is a Lie algebra with respect to the operation of standard commutator. Show that

$$\mathrm{span}\langle \frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\rangle$$

is a Lie subalgebra of D. Prove that this subalgebra is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

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