PROBLEM SET 10, DUE NOVEMBER 29

10. Lie Groups and Lie Algebras Representations

Problem 10.1. *Find the Lie algebra of the Lie group $Sp_n(\mathbb{F})$ and its dimension.*

Problem 10.2. *Show that the map* $\exp : \mathfrak{su}_2 \to SU_2$ *is not injective.*

Problem 10.3. *Show that the map* $\exp : \mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ *is not surjective.*

Problem 10.4. *Show that*

$$\mathrm{Sp}(1,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}), \quad \mathrm{Sp}(1,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C}).$$

Problem 10.5. *Show that*

$$\rho(E)v_k = k(\mu - k + 1)v_{k-1}, \quad 1 \le k \le n,$$

in the proof of Proposition 1.4 in Lecture 10.

Problem 10.6. *Is the Heisenberg Lie algebra* $\mathfrak{uut}(3,\mathbb{C})$ *isomorphic to the Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$*?*

Problem 10.7. *Are the Lie algebras* $\mathfrak{su}(2)$ *and* $\mathfrak{so}(3,\mathbb{R})$ *isomorphic?*

Problem 10.8. *Is the standard representation of* $\mathfrak{sl}_n(\mathbb{C})$ *in the vector space* $\mathbb{C}^n$ *irreducible?*

Problem 10.9. *Is the standard representation of* $\mathfrak{so}_n(\mathbb{C})$ *in the vector space* $\mathbb{C}^n$ *irreducible?*

Problem 10.10. *Is the standard representation of* $\mathfrak{ut}_n(\mathbb{C})$ *in the vector space* $\mathbb{C}^n$ *irreducible?*

Problem 10.11. *Is the adjoint representation of* $\mathfrak{gl}_n(\mathbb{C})$ *in the vector space* $\mathfrak{gl}_n(\mathbb{C})$ *irreducible?*

Problem 10.12. *Is the adjoint representation of* $\mathfrak{sl}_n(\mathbb{C})$ *in the vector space* $\mathfrak{sl}_n(\mathbb{C})$ *irreducible?*

Problem 10.13. *Consider the restriction R of the adjoint representation of the Lie algebra* $\mathfrak{sl}_3(\mathbb{C})$ *to its Lie subalgebra* $\mathfrak{sl}_2(\mathbb{C})$*. Find the multiplicity of each irreducible* $\mathfrak{sl}_2(\mathbb{C})$*-module* τ_n *(with the highest weight n) in the decomposition of R as a sum of irreducible representations of* $\mathfrak{sl}_2(\mathbb{C})$*. You may assume that every representation of the Lie algebra* $\mathfrak{sl}_2(\mathbb{C})$ *is completely reducible if you need.*

Date: November 23, 2025.