PROBLEM SET 11, DUE DECEMBER 13

11. Lie Groups and Lie Algebras Representations, part 2

In the following problems we may take for granted that every representation of the Lie group $SL_2(\mathbb{C})$ or the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is completely reducible.

Problem 11.1. *Let T be an irreducible representation of a matrix Lie group G. Is its tangent representation dT necessarily irreducible?*

Problem 11.2. *Show that the representation dT_n of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ in the space of homogeneous polynomials of degree n is irreducible.*

Problem 11.3. *Find an explicit isomorphism between the representation dT_n and the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module with the highest weight n by describing the preimages of the basis vectors $v_0, v_1, \dots, v_n$.*

Problem 11.4. *Denote by $Ad : SU_2 \to \mathrm{GL}(\mathfrak{su}_2)$ the representation defined by $Ad(A)(X) := AXA^{-1}$. An arbitrary element of the Lie algebra $\mathfrak{su}_2$ can be represented as*

$$\begin{pmatrix} ix_1 & x_2+ix_3 \\ -x_2+ix_3 & -ix_1 \end{pmatrix}, x_1, x_2, x_3 \in \mathbb{R}$$

It follows that $\det(X) = x_1^2+x_2^2+x_3^2$ is preserved by the map Ad. Thus, $\mathrm{im}(Ad) \subset O(\mathfrak{su}_2) \simeq O_3(\mathbb{R})$. Show that

$$\mathrm{im}(Ad) = SO(\mathfrak{su}_2) \simeq SO_3(\mathbb{R})$$

and determine the kernel.

Problem 11.5. *Denote by $Ad : SU_2 \to \mathrm{GL}(\mathfrak{su}_2)$ the representation defined via $Ad(A)(X) = AXA^{-1}$. Describe the map $d(Ad)$ in some basis.*

Problem 11.6. *Let V_n be the complex vector space of homogeneous polynomials of degree n in variables x and y. Recall the definition of the representation $T_n : SL_2(\mathbb{C}) \to GL(V_n)$ by*

$$\left(\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)P\right)(x,y) := P(ax+cy, bx+dy).$$

Is the representation $S^n(T_1)$ isomorphic to the representation T_n ?

Problem 11.7. *Denote by $\tau_n : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_n)$ the representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ corresponding to the highest weight module with the highest weight n. Find the decomposition of the representation $\tau_n \otimes \tau_m$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into the sum of its irreducible representations.*

Problem 11.8. *Let V_n be the complex vector space of homogeneous polynomials of degree n in the variables x and y. Define the map $T_n : SL_2(\mathbb{C}) \to GL(V_n)$ by*

$$\left(\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)P\right)(x,y) := P(ax+cy, bx+dy). \tag{11.1}$$

Find the decomposition of the representation $T_n \otimes T_m$ of the matrix Lie group $SL_2(\mathbb{C})$ into a sum of its irreducible representations. Hint: We can compute the traces of the representation T_n and $T_n \otimes T_m$ at the element

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \tag{11.2}$$

and find the conditions on the multiplicities.

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Problem 11.9. *Decompose the representation $\Lambda^2(\tau_4)$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into the sum of its irreducible representations.*

Problem 11.10. *Decompose the representation $S^2(\tau_4)$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ into the sum of its irreducible representations.*

Problem 11.11. *Denote by St the standard representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Is the representation $\Lambda^2 St$ irreducible?*

Problem 11.12. *Is the representation $\Lambda^2(ad_{\mathfrak{so}_3(\mathbb{R})})$ of the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ irreducible?*

Problem 11.13. *Let ρ be a representation of the Lie algebra $sl_2(\mathbb{C})$. The endomorphism*

$$K_\rho = \rho(E)\rho(F) + \rho(F)\rho(E) + \rho(H)^2/2$$

is called a Casimir element.

Show that K_ρ commutes with the action of $sl_2(\mathbb{C})$.

Problem 11.14. *Find K_ρ when ρ is the highest weight $\mathfrak{sl}_2(\mathbb{C})$-module τ_n.*