LINEAR ALGEBRA, PART 1. VECTOR SPACES, BASES.

Throughout this lecture, we will assume that $\mathbb{F}$ is a field and V is a vector space over $\mathbb{F}$. Please treat each proposition in this text as an exercise.

1. Definition of a Vector Space

Definition 1.1. Suppose that $\mathbb{F}$ is a field whose elements we call scalars. A non-empty set V with an operation of addition and an operation of scalar multiplication $\text{scal} : \mathbb{F} \times V \to V\ ((\lambda, x) \mapsto \lambda x)$ is called a vector space over $\mathbb{F}$ if the following conditions hold:

(1) $(V, +)$ is an abelian group. We denote its identity element by 0 or by 0_V.
(2) $\lambda(x + y) = \lambda x + \lambda y, \quad \forall \lambda \in \mathbb{F}, \forall x, y \in V.$
(3) $(\lambda + \mu)x = \lambda x + \mu x, \quad \forall \lambda, \mu \in \mathbb{F}, \forall x \in V.$
(4) $(\lambda\mu)x = \lambda(\mu x), \quad \forall \lambda, \mu \in \mathbb{F}, \forall x \in V.$
(5) $1_{\mathbb{F}} x = x, \quad \forall x \in V.$

The notation here may be a bit confusing. For example, in condition (3), the symbol "+" in the LHS stands for addition in the field $\mathbb{F}$, and in the RHS, it stands for addition in V.

Definition 1.2. For a field $\mathbb{F}$ denote by

$$\mathbb{F}^n := \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_j \in \mathbb{F}, \forall 1 \leq j \leq n \right\} \tag{1.1}$$

the set of ordered n-tuples of elements of $\mathbb{F}$. We define component-wise addition on $\mathbb{F}^n$ and multiplication of elements of $\mathbb{F}^n$ by scalars of $\mathbb{F}$:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \forall a_j, b_j \in \mathbb{F}; \tag{1.2}$$

$$\lambda \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}, \quad \forall a_j, \lambda \in \mathbb{F}. \tag{1.3}$$

2. Subspaces, Direct Sums, Quotient Spaces

Definition 2.1. Let V be a vector space over a field $\mathbb{F}$. A non-empty subset U of V is called a subspace of V if:

(1) for every pair of elements $x, y \in U$, their sum $x + y$ belongs to U;
(2) for every scalar $\lambda \in \mathbb{F}$ and every element $x \in U$, the product λx belongs to U.

Definition 2.2. A non-empty subset U of a vector space V is called a subspace of V if U is a vector space over $\mathbb{F}$ with respect to the addition and multiplication by scalars defined in V.

Exercise 2.3. Show that the two preceding definitions are equivalent.

It is widely agreed that if V is a vector space, then the notation $U \subset V$ means that U is a subspace of V unless specified otherwise.

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Definition 2.4. Suppose that $U_1, \dots, U_n$ are subspaces of the vector space V. Define a set

$$\sum_{j=1}^{n} U_j := \left\{ \sum_{j=1}^{n} x_j \mid x_j \in U_j, \forall 1 \le j \le n \right\}. \tag{2.1}$$

It is called the sum or the join of the subspaces $U_1, \dots, U_n$. Another notation for it is $U_1 + U_2 + \dots + U_n$.

Definition 2.5. Suppose that $U_1, \dots, U_n$ are subspaces of the vector space V. We write

$$V = \bigoplus_{j=1}^{n} U_j \tag{2.2}$$

and say that V is a direct sum of $U_1, \dots, U_n$ if and only if $\forall x \in V \, \exists! (x_1, \dots, x_n)$ such that

(1) $x_j \in U_j$ for all $1 \le j \le n$;
(2) $x = \sum_{j=1}^{n} x_j$.

Definition 2.6. Let U be a subspace of a vector space V. Then we say that $x, y \in V$ are equivalent if and only if $x - y \in U$. We write

$$(x \sim_U y) \iff (x - y \in U). \tag{2.3}$$

Exercise 2.7. Show that $\sim_U$ is indeed an equivalence relation for every subspace U.

Exercise 2.8. Let U be a subspace of a vector space V over $\mathbb{F}$. Consider equivalence classes in V with respect to the relation $\sim_U$. If a class contains an element x, we may denote that class by $x + U$. We define the operations of addition and multiplication by scalars on these equivalence classes by

$$(x + U) + (y + U) := (x + y) + U, \quad \lambda(x + U) := (\lambda x) + U. \tag{2.4}$$

Show that the operations are well-defined; in other words, the results do not depend on the choices of representatives. Also, show that these operations turn the set of equivalence classes into a vector space over $\mathbb{F}$. It is called a quotient space and is denoted by V/U.

3. Linear Span and Linear Independence

Definition 3.1. For a list of vectors $(v_1, \dots, v_n)$ we define its linear span to be the set

$$\boxed{\langle v_1, \dots, v_n \rangle := \left\{ \sum_{j=1}^{n} a_j v_j \mid a_1, \dots, a_n \in \mathbb{F} \right\}}. \tag{3.1}$$

The expression $\sum_{j=1}^{n} a_j v_j$ is called a linear combination of the vectors $v_1, \dots, v_n$. We consider a list of vectors rather than a set to allow a vector to appear more than once.

We allow a list to be empty. In that case,

$$\langle \emptyset \rangle = \{0\}. \tag{3.2}$$

Definition 3.2. An ordered list of vectors $(v_1, \dots, v_n)$ is called linearly dependent if there exists $(a_1, \dots, a_n) \in \mathbb{F}^n$ such that:

(1) $(a_1, \dots, a_n) \neq (0, \dots, 0)$;
(2)

$$\sum_{j=1}^{n} a_j v_j = 0. \tag{3.3}$$

A list of vectors is called linearly independent if it is not linearly dependent.

A linearly independent list obviously forms a set.

Proposition 3.3. ***(Linear Dependence Lemma)*** *Suppose that a list of vectors $(u_1, \ldots, u_m)$ is linearly dependent, and $u_1 \neq 0$. Show that there exists j, with $2 \leq j \leq m$, such that the following hold:*

(1) $u_j \in \langle u_1, \ldots, u_{j-1} \rangle$;
(2) *if the j-th term is removed from $(u_1, \ldots, u_m)$, then the span of the remaining list equals $\langle u_1, \ldots, u_m \rangle$.*

Now, we can prove a crucial result that will allow us to introduce the notion of the dimension of a vector space later.

Proposition 3.4. ***(Replacement Theorem)*** *Suppose that*

$$\langle v_1, \ldots, v_n \rangle = V, \tag{3.4}$$

and a subset $\{u_1, \ldots, u_m\}$ of V is linearly independent. Then $m \leq n$.

4. A Basis of a Vector Space

Definition 4.1. A set of vectors $\{v_1, \ldots, v_n\}$ is called a basis of the vector space V if the following two conditions hold:

(1) The set $\{v_1, \ldots, v_n\}$ is linearly independent;
(2) $\langle v_1, \ldots, v_n \rangle = V$.

Let us consider other equivalent ways to look at the notion of a basis.

Proposition 4.2. *Let $\{v_1, \ldots, v_n\}$ be a set of vectors in V. The following conditions are equivalent:*

(1) *$\{v_1, \ldots, v_n\}$ is a basis of V.*

(2) *$\forall x \in V \; \exists! (a_1, \ldots, a_n) \in \mathbb{F}^n$ such that*

$$x = \sum_{j=1}^{n} a_j v_j. \tag{4.1}$$

(3)

$$V = \bigoplus_{j=1}^{n} \langle v_j \rangle. \tag{4.2}$$

(4) *$\{v_1, \ldots, v_n\}$ is a minimal spanning set of V, that is, $\{v_1, \ldots, v_n\}$ spans V, but any proper subset of $\{v_1, \ldots, v_n\}$ does not span V.*

(5) *$\{v_1, \ldots, v_n\}$ is a maximal linearly independent set, that is, $\{v_1, \ldots, v_n\}$ is linearly independent, but after adding any vector to this set, it becomes linearly dependent.*

Let us list corollaries from Proposition 3.4.

Proposition 4.3. *Let $\{v_1, \ldots, v_n\}$ be a basis of V. Then*

(1) *Every list of vectors from V that has length greater than n is linearly dependent;*

(2) *If $\langle u_1, \ldots, u_m \rangle = V$, then $m \geq n$;*

(3) *If* $\langle w_1, \ldots, w_n \rangle = V$, *then the set* $\{w_1, \ldots, w_n\}$ *is a basis of* V;

(4) *If a set* $\{x_1, \ldots, x_n\}$ *is linearly independent, then it is a basis of the vector space* V;

(5) *If a set* $\{y_1, \ldots, y_m\}$ *is a basis of* V, *then* $m = n$.

5. Dimension of a Vector Space and Its Properties

Definition 5.1. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Then we say that V is a finite-dimensional space, and the corresponding notation is $\dim V < \infty$. The number n is called the dimension of the vector space V. In this case, we write $\dim V = n$.

Note that $\dim(\{0\}) = 0$.

Next, we discuss how we can construct a basis from existing subsets of vectors (Proposition 5.2 and Exercise 5.5).

Proposition 5.2. *Suppose that* $\dim V = n$ *and* $\{u_1, \ldots, u_m\}$ *is a linearly independent subset of* V. *Show that there exist vectors* $w_1, \ldots, w_{n-m}$ *such that* $\{u_1, \ldots, u_m, w_1, \ldots, w_{n-m}\}$ *is a basis of* V.

Proposition 5.3. *Let* U *be a subspace of a finite-dimensional vector space* V. *Then there exists a subspace* W *such that*

$$V = U \oplus W. \tag{5.1}$$

Proposition 5.4. *Let* U *be a subspace of a finite-dimensional vector space* V. *Then the quotient space* V/U *is also finite-dimensional and*

$$\dim(V/U) = \dim V - \dim U. \tag{5.2}$$

Proposition 5.5. *Suppose that* $\dim V = n$ *and*

$$\langle u_1, \ldots, u_m \rangle = V. \tag{5.3}$$

Then there exists a subset of $\{u_1, \ldots, u_m\}$ *that is a basis of* V.

Proposition 5.6. *Suppose* U *and* W *are subspaces of a finite-dimensional vector space* V. *Then*

$$\boxed{\dim(W + U) = \dim W + \dim U - \dim(W \cap U).} \tag{5.4}$$

Proposition 5.7. *Suppose that* $\dim V < \infty$ *and*

$$V = \bigoplus_{j=1}^{k} U_j. \tag{5.5}$$

Then

$$\dim V = \sum_{j=1}^{k} \dim U_j. \tag{5.6}$$

6. Exercises

Exercise 6.1. Suppose that U_1, U_2 and U_3 are subspaces of a vector space V. Is it true that

$$V = U_1 \oplus U_2 \oplus U_3$$

if and only if the following conditions hold?

(1) $V = U_1 + U_2 + U_3$;
(2) $U_1 \cap U_2 = \{0\}$;
(3) $U_1 \cap U_3 = \{0\}$;
(4) $U_2 \cap U_3 = \{0\}$.

Exercise 6.2. Find a basis for the subspace

$$\{p \in \mathbb{R}_n[x] \mid p(0) = 0, p(1) = 0\} \subset \mathbb{R}_n[x],$$

where $\mathbb{R}_n[x]$ is the space of polynomials with real coefficients of degree not greater than n.

Exercise 6.3. a) Does there exist a basis $\{p_1, p_2, p_3, p_4\}$ of $\mathbb{R}_3[x]$ such that none of the polynomials p_1, p_2, p_3, p_4 has degree 3?

b) Does there exist a basis $\{p_1, p_2, p_3, p_4\}$ of $\mathbb{R}_3[x]$ such that none of the polynomials p_1, p_2, p_3, p_4 has degree 2?

Exercise 6.4. Denote by $\mathrm{Mat}(n \times n, \mathbb{R})$ the vector space of real-valued $n \times n$ matrices, and by U the subspace of symmetric matrices.

a) Find a basis of U.

b) Find a subspace W such that

$$\mathrm{Mat}(n \times n, \mathbb{R}) = U \oplus W.$$

Chapters from textbooks relevant to this lecture:

- Friedberg, Insel, Spence, Sections 1.2–1.7;
- Axler, Chapters 1 and 2.