LINEAR ALGEBRA, PART 2. LINEAR MAPS. MATRICES.

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and all vector spaces are defined over that same field. Please treat every proposition in this lecture as an exercise.

1. Types of Linear Maps

Definition 1.1. Suppose that V and U are vector spaces over the same field $\mathbb{F}$. A map $A : V \to U$ is called a linear map (or, a linear transformation, or a homomorphism) if

$$\boxed{A(x+y) = Ax + Ay, \quad A(\lambda x) = \lambda A x, \quad \forall x, y \in V, \forall \lambda \in \mathbb{F}.} \tag{1.1}$$

The set of all linear maps from V to U is denoted by $\mathrm{Hom}(V, U)$.

Definition 1.2. We may define pointwise addition and multiplication by scalars in $\mathrm{Hom}(V, U)$:

$$(A+B)(x) := Ax + Bx, \quad (\lambda A)(x) := \lambda(Ax), \quad \forall A, B \in \mathrm{Hom}(V, U), \forall x \in V, \forall \lambda \in \mathbb{F}. \tag{1.2}$$

Exercise 1.3. Show that if $A, B \in \mathrm{Hom}(V, U)$ then $A + B, \lambda A \in \mathrm{Hom}(V, U)$.

Exercise 1.4. Show that $\mathrm{Hom}(V, U)$ is a vector space with respect to those operations.

Definition 1.5. We are going to use the following terms and notation:

(1) A linear map from a vector space to itself is called an endomorphism or a linear operator. The set of all linear operators on V is denoted by $\mathrm{End}(V)$. A map $\mathrm{Id} \in \mathrm{End}(V)$ defined by
$$\mathrm{Id}(x) := x, \quad \forall x \in V \tag{1.3}$$
is called the identity map.
(2) A linear map is called a monomorphism or an embedding if it is injective.
(3) A linear map is called an epimorphism if it is surjective.
(4) A linear map is called an isomorphism if it is bijective. The set of all isomorphisms from a vector space V to a vector space U is denoted by $\mathrm{Iso}(V, U)$. If there exists an isomorphism from V to U, we will write $V \cong U$.
(5) A linear operator is called an automorphism if it is bijective. The set of all automorphisms of a vector space V is denoted by $\mathrm{GL}(V)$ (which stands for "General Linear (Group)") or by $\mathrm{Aut}(V)$.

Definition 1.6. Suppose that V, U, W are finite-dimensional spaces, $A \in \mathrm{Hom}(V, U), B \in \mathrm{Hom}(U, W)$. We may also consider the operation of composition of linear maps B and A. We are going to denote this composition of linear maps by BA:

$$BA(x) := B(A(x)), \quad \forall x \in V. \tag{1.4}$$

We use this notation because the operation of the composition of linear operators shares several nice properties with multiplication.

Date: August 11, 2025.

2. Image and Kernel of a Linear Map

Definition 2.1. For an arbitrary $A \in \mathrm{Hom}(V,U)$ we put

$$\ker(A) := \{x \in V \mid Ax = 0\}, \quad \mathrm{im}(A) := \{y \in U \mid \exists x \in V \text{ such that } Ax = y\}. \tag{2.1}$$

The set $\ker(A)$ is called the kernel of A. The set $\mathrm{im}(A)$ is called the image of A.

Proposition 2.2. *For an arbitrary $A \in \mathrm{Hom}(V,U)$ the set $\ker(A)$ is a subspace of the vector space V, $\mathrm{im}(A)$ is a subspace of the vector space U.*

Exercise 2.3. Show that

$$(A \text{ is a monomorphism}) \iff (\ker A = \{0\}). \tag{2.2}$$

Next, we will introduce a version of the fundamental theorem on homomorphisms for vector spaces.

Proposition 2.4. *(The First Isomorphism Theorem) For an arbitrary $A \in \mathrm{Hom}(V,U)$ define the map p from the quotient space $V/\ker(A)$ to the vector space U by*

$$p(x + \ker(A)) := Ax, \quad \forall x \in V. \tag{2.3}$$

Then, p is a well-defined linear map. Moreover, p is an isomorphism between the quotient space $V/\ker(A)$ and $\mathrm{im}(A)$.

Definition 2.5. Suppose that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V,U)$ we put

$$\mathrm{rk}(A) := \dim(\mathrm{im}(A)). \tag{2.4}$$

The number $\mathrm{rk}(A)$ is called the rank of the homomorphism A, the number $\dim(\ker(A))$ is called its nullity.

Proposition 2.6. *(The Rank Plus Nullity Theorem) Suppose that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V,U)$ we get the identity*

$$\boxed{\mathrm{rk}(A) + \dim(ker(A) = \dim V.} \tag{2.5}$$

3. Isomorphisms

The following two propositions will provide another way to look at isomorphisms.

Proposition 3.1. *A linear map $A \in \mathrm{Hom}(V,U)$ is an isomorphism if and only if $\ker(A) = \{0\}$ and $\mathrm{im}(A) = U$. Then the inverse map A^{-1} is also linear and, therefore, $A^{-1} \in \mathrm{Iso}(U,V)$.*

Proposition 3.2. *Suppose that V and U are vector spaces and $A \in \mathrm{Hom}(V,U)$. Then*

$$(A \in \mathrm{Iso}(V,U)) \iff (\exists B \in \mathrm{Hom}(U,V) \text{ such that } AB = BA = \mathrm{Id}). \tag{3.1}$$

If we can pick a finite basis in a vector space, then we may construct an isomorphism of that space with one of the standard vector spaces $\mathbb{F}^n$.

Proposition 3.3. *Suppose that V is a finite-dimensional space, $\alpha = (v_1, \ldots, v_n)$ is a basis of V. We assume that the basis is ordered, i.e., the order of the elements in the set matters. We use parentheses instead of curly brackets to emphasize that we consider an ordered sequence.*

As we know, $\forall x \in V\ \exists!(a_1, \ldots, a_n) \in \mathbb{F}^n$ such that

$$x = \sum_{j=1}^{n} a_j v_j. \tag{3.2}$$

The numbers $(a_1, \dots, a_n)$ are called the coordinates of the vector x in the basis α. Let us define a map $\varphi_\alpha : V \to \mathbb{F}^n$ by setting

$$\varphi_\alpha(x) := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{3.3}$$

Then the map $\varphi_\alpha \in \mathrm{Iso}(V, \mathbb{F}^n)$.

That claim allows us to provide the following classification of finite-dimensional vector spaces over the same field up to an isomorphism.

Proposition 3.4. *Suppose that V and U are finite-dimensional vector spaces. There exists an isomorphism between V and U (or, we may say that V and U are isomorphic) if and only if $\dim V = \dim U$.*

Proposition 3.5. *Suppose that V and U are finite-dimensional vector spaces and $\dim V = \dim U$.*
a) If $A \in Hom(V, U)$ is a monomorphism, then A is an isomorphism.
b) If $A \in \mathrm{Hom}(V, U)$ is an epimorphism, then A is an isomorphism.

4. The Matrix of a Linear Map

Proposition 4.1. *Suppose that $(v_1, \dots, v_n)$ is a basis of a vector space V.*
a) Then for every list of vectors $y_1, \dots, y_n \in U$ there exists a unique map $B \in \mathrm{Hom}(V, U)$ such that $Bv_j = y_j$, $\forall 1 \le j \le n$.
b) Every map $A \in \mathrm{Hom}(V, U)$ is completely determined by the ordered set of images $Av_1, \dots, Av_n$.

The next definition establishes how we can represent linear maps via matrices.

Definition 4.2. Suppose that $\alpha = (v_1, \dots, v_n)$ is an ordered basis of a vector space V, $\beta = (u_1, \dots, u_m)$ is an ordered basis of a vector space U, $A \in \mathrm{Hom}(V, U)$. Then there exists a list of numbers (a_i^j) such that

$$\boxed{Av_i = \sum_{j=1}^{m} a_i^j u_j, \quad \forall 1 \le i \le n.} \tag{4.1}$$

The matrix filled out with (a_i^j)

$$\begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{pmatrix} \tag{4.2}$$

is called the matrix of the map A (in the bases α and β) and is denoted by the same letter A or by $[A]_\alpha^\beta$. The set of $m \times n$ matrices with the entries from the field $\mathbb{F}$ is denoted by $\mathrm{Mat}(m \times n, \mathbb{F})$.

Exercise 4.3. Suppose that V and U are finite-dimensional vector spaces with a fixed ordered basis in each of them, $A, B \in \mathrm{Hom}(V, U)$. Express the entries of the matrices of the linear maps $A + B$ and λA using the entries of A and B.

Exercise 4.4. It is obvious that the set $\mathrm{Mat}(m \times n, \mathbb{F})$ is a vector space with respect to entrywise addition and entrywise multiplication by scalars. Suppose that V and U are finite-dimensional vector spaces. Show that the vector spaces $\mathrm{Hom}(V, U)$ and $Mat(\dim U \times \dim V, \mathbb{F})$ are isomorphic.

Exercise 4.5. Suppose that V and U are finite-dimensional vector spaces. Show that

$$\boxed{\dim(\mathrm{Hom}(V, U) = \dim V \cdot \dim U.} \tag{4.3}$$

Now, we will establish the rule of multiplication of matrices that corresponds to the composition of linear maps.

Definition 4.6. (The Row-Column rule) Suppose that V, U, W are finite-dimensional spaces with bases α, β, γ respectively, $A \in \mathrm{Hom}(V, U), B \in \mathrm{Hom}(U, W)$. We define the product of matrices via the formula

$$\boxed{[B]^\gamma_\beta \cdot [A]^\beta_\alpha := [BA]^\gamma_\alpha.} \tag{4.4}$$

Exercise 4.7. Suppose that X is an arbitrary $m \times n$ matrix, Y is an arbitrary $p \times m$ matrix. Show that Definition 4.6 defines a unique product $Y \cdot X$, which is an $p \times n$ matrix.

5. Change of Bases for Linear Maps

Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V. Recall that in the previous lecture, we introduced the notation $\varphi_\alpha(x)$ for the linear map from V to $\mathbb{F}^n$ assigning to a vector x its coordinates in a basis α. Also, we used the notation $[A]^\beta_\alpha$ for the matrix of a linear map A expressing the images of the vectors of the basis α as linear combinations of the vectors of the basis β. In our current notation, it takes the following form:

$$Av_i = \sum_{j=1}^{n} a^j_i v'_j, \quad \forall 1 \le i \le n, \quad [A]^\beta_\alpha = (a^j_i)_{1 \le i,j \le n}. \tag{5.1}$$

Before defining the change-of-basis matrix, we will establish some of its properties.

Proposition 5.1. *Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V.*

a) Then the matrix $[\mathrm{Id}]^\beta_\alpha$ is invertible.

b) The matrix $[\mathrm{Id}]^\beta_\alpha$ coincides with the matrix of the linear operator $\varphi_\beta \circ (\varphi_\alpha)^{-1}$ in the standard basis.

Definition 5.2. Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V. The matrix $[\mathrm{Id}]^\beta_\alpha$ is called the change of basis matrix from α to β.

From Proposition 5.1, we see that the coordinates of a vector in one basis can be obtained from its coordinates in the other by applying the change-of-basis matrix:

$$\varphi_\beta(x) = [\mathrm{Id}]^\beta_\alpha \varphi_\alpha(x). \tag{5.2}$$

Recall the following rule connecting the composition of linear maps with the matrix multiplication.

$$[B]^\gamma_\beta \cdot [A]^\beta_\alpha := [BA]^\gamma_\alpha. \tag{5.3}$$

We may use it to establish the following claim.

Proposition 5.3. *Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V, $\gamma = (u_1, \ldots, u_m)$, and $\delta = (u'_1, \ldots, u'_m)$ are two ordered bases of a vector space U, C_1 is the change of basis matrix from α to β, C_2 is the change of basis matrix from γ to δ, and $A \in \mathrm{Hom}(V, U)$. Then the matrices $[A]^\gamma_\alpha$ and $[A]^\delta_\beta$ satisfy the following relation:*

$$\boxed{[A]^\delta_\beta = C_2\, [A]^\gamma_\alpha\, C_1^{-1}.} \tag{5.4}$$

Let us consider the case of linear operators. We agree that for an operator $A \in \mathrm{End}(V)$, we pick a single basis α in V. So the matrices of endomorphisms A have the form $[A]^\alpha_\alpha$ for some basis α. Then, we may reformulate Proposition 5.3 for that case.

Proposition 5.4. *Suppose that $\alpha = (v_1, \dots, v_n)$ and $\beta = (v'_1, \dots, v'_n)$ are two ordered bases of a vector space V, C is the change of basis matrix from α to β, and $A \in \mathrm{End}(V)$. Then the matrices $[A]^\alpha_\alpha$ and $[A]^\beta_\beta$ satisfy the following relation:*

$$\boxed{[A]^\beta_\beta = C[A]^\alpha_\alpha C^{-1}.} \tag{5.5}$$

6. Equivalence of Matrices

Following the result of Proposition 5.3, we define the equivalence of matrices.

Definition 6.1. Suppose that $A, B \in \mathrm{Mat}(m \times n, \mathbb{F})$. They are called equivalent if there exist invertible matrices $C \in \mathrm{Mat}(m \times m, \mathbb{F})$, $D \in \mathrm{Mat}(n \times n, \mathbb{F})$ such that

$$A = C\,B\,D. \tag{6.1}$$

The analogous notion for linear operators is called similarity of square matrices.

Definition 6.2. Suppose that $A, B \in \mathrm{Mat}(n \times n, \mathbb{F})$. They are called similar if there exists an invertible matrix $C \in \mathrm{Mat}(n \times n, \mathbb{F})$ such that

$$A = C\,B\,C^{-1}. \tag{6.2}$$

Using the ideas behind the First Isomorphism Theorem, we can essentially solve the problem of classification of equivalent matrices.

Proposition 6.3. *Suppose that V and U are finite-dimensional vector spaces, $A \in \mathrm{Hom}(V, U)$, and $\mathrm{rk}(A) = k$. Then there exist bases α and β of the spaces V and U, respectively, such that the matrix of the linear map has the following block diagonal form:*

$$\boxed{[A]^\beta_\alpha = \begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & 0 \end{pmatrix},} \tag{6.3}$$

where Id_k is the $k \times k$ identity matrix.

Therefore, we see that every equivalence class of matrices of the same size consists of the matrices of the same rank.

The problem of classification of similar matrices turns out to be more complicated.

7. System of Linear Equations

A system of linear equations

$$\sum_{i=1}^{n} a^j_i x_i = b_j, \quad 1 \le j \le m \tag{7.1}$$

can be represented as $Ax = b$, where

$$A = (a^j_i)_{1 \le i \le n,\, 1 \le j \le m}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}. \tag{7.2}$$

This notation allows us to formulate two general statements about those systems. We are going to view A both as a matrix and as a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$ defined by that matrix in the standard bases.

Proposition 7.1. *Solutions of the system $Ax = 0$ form a subspace of the vector space $\mathbb{F}^n$. The dimension of the subspace of solutions of the system $Ax = 0$ equals $n - \mathrm{rk}(A)$. In particular, if $m < n$, then there exists a non-zero solution to the homogeneous system.*

Proposition 7.2. *A solution of the system $Ax = b$ exists if and only if $b \in \mathrm{im}A$. In particular, if $\mathrm{rk}(A) = m$, then a solution exists. The difference between any two solutions of the system $Ax = b$ is itself a solution of the system $Ax = 0$.*

8. Dual Space

Definition 8.1. Let V be a vector space over a field $\mathbb{F}$. The vector space $\mathrm{Hom}(V, \mathbb{F})$ is called the dual space (to the vector space V) and is denoted by V^*. The elements of V^* are called linear functionals, or covectors, or linear forms.

Definition 8.2. Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V. Define $f^i \in V^*$ at the basis vectors via

$$\boxed{f^i(v_j) = \delta^i_j, \quad 1 \leq i, j \leq n,} \tag{8.1}$$

where δ^i_j is the Kronecker delta:

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{8.2}$$

The set $\{f^1, f^2, \ldots, f^n\}$ is called the dual basis (to the basis $\{v_1, \ldots, v_n\}$).

Exercise 8.3. Show that $\{f^1, f^2, \ldots, f^n\}$ is a basis for V^*.

Proposition 8.4. *Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$. Then there exists an isomorphism of V and $(V^*)^*$ such that it does not depend on the choice of basis in V.*

Exercise 8.5. Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V and C is the change of basis matrix from α to β. Denote by γ and δ the bases of V^* dual to α and β, respectively. Express the change-of-basis matrix from γ to δ in terms of the matrix C.

Definition 8.6. Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{Hom}(V, U)$. Define $A^* \in \mathrm{Hom}(U^*, V^*)$ by

$$\boxed{(A^*(f))(v) = f(Av), \quad \forall v \in V, \forall f \in U^*.} \tag{8.3}$$

A^* is called the operator adjoint of A.

Exercise 8.7. Show that for any $A \in \mathrm{Hom}(V, U)$ the map $A^* \in \mathrm{Hom}(U^*, V^*)$ exists and it is unique.

Exercise 8.8. Suppose that V and U are finite-dimensional vector spaces, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, $\gamma = \{f^1, \ldots, f^n\}$ is its dual basis, $\beta = \{u_1, \ldots, u_m\}$ is a basis of U, $\delta = \{g^1, \ldots, g^m\}$ is its dual basis, $A \in \mathrm{Hom}(V, U)$ and its matrix $[A]^\beta_\alpha$ equals

$$(a^j_i)_{1 \leq i \leq n, 1 \leq j \leq m} \tag{8.4}$$

Find the matrix of $[A^*]^\gamma_\delta$.

9. Exercises

Exercise 9.1. Suppose that V and U are finite-dimensional spaces. For which subspaces W of the space V there exist $A \in \mathrm{Hom}(V, U)$ such that $\ker A = W$?

Exercise 9.2. Let A be $m \times n$ matrix. The linear span of its rows is a subspace of the vector space $\mathbb{F}^n$, the linear span of its columns is a subspace of the vector space $\mathbb{F}^m$. Show that the dimensions of those two subspaces are equal.

Exercise 9.3. Let V be a finite-dimensional space over a field $\mathbb{F}$. Assume that $A \in End(V)$ such that

$$AB = BA, \quad \forall B \in \mathrm{End}(V).$$

Does there always exist $\lambda \in \mathbb{F}$ such that $A = \lambda \mathrm{Id}$?

Exercise 9.4. Suppose that P is a linear operator in a vector space V. Show that $P^2 = P$ if and only if the following two conditions hold:

(1) $V = \ker P \oplus \mathrm{im} P$;
(2) $Px = x, \quad \forall x \in \mathrm{im} P$.

P is called a projection operator.

Exercise 9.5. Let U be a subspace of a finite-dimensional space V. Define the annihilator of U by

$$\mathrm{ann}(U) := \{f \in V^* \mid f(u) = 0, \forall u \in U\}.$$

Show that $\mathrm{ann}(U)$ is a subspace of V^*. Express $\dim(\mathrm{ann}(U))$ using $\dim V$ and $dim U$.

Exercise 9.6. Suppose that A is a linear operator in a finite-dimensional vector space V. Show that

$$\ker(A^*) = \mathrm{ann}(\mathrm{im}(A)).$$

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 2.1 – 2.6;
- Axler, ch.3;
- Roman, pp.59 – 79, 94 – 101.