LINEAR ALGEBRA, PART 3. TENSOR PRODUCTS OF VECTOR SPACES

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please consider proof of every proposition in this lecture as an exercise.

1. Tensor Products of Vector Spaces

Definition 1.1. Suppose that V and U are vector spaces over a field $\mathbb{F}$. Denote by $Free(V,U)$ the set of all formal finite linear combinations of elements of $V \times U$:

$$Free(V,U) := \left\{ \sum_{i=1}^{n} a_i(x_i, y_i) \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\}. \tag{1.1}$$

(Note that the pairs (x_i, y_i) are simply symbols here. It might be less confusing to write $Free(V,U)$ as

$$\left\{ \sum_{i=1}^{n} a_i e_{(x_i,y_i)} \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\} \tag{1.2}$$

but this would lead to more cumbersome expressions later in the definition.)

The set $Free(V,U)$ has a natural structure of a vector space. We now may define a subspace $Ident(V,U) \subset Free(V,U)$ as

$$\begin{aligned} Ident(V,U) = \operatorname{span}\langle \{(ax+bz,y) - a(x,y) - b(z,y) \,|\, x,z \in V, y \in U, a,b \in \mathbb{F}\} \cup \\ \{(x, ay+bt) - a(x,y) - b(x,t) \,|\, x \in V, y,t \in U, a,b \in \mathbb{F}\} \rangle. \end{aligned} \tag{1.3}$$

The tensor product of vector spaces V and U is, by definition, the quotient of $Free(V,U)$ by $Ident(V,U)$. We denote the tensor product of vector spaces V and U by $V \otimes U$. Also, we denote by

$$\sum_{i=1}^{n} a_i x_i \otimes y_i \tag{1.4}$$

the equivalence class that contains the element $\sum_{i=1}^{n} a_i(x_i, y_i)$. A map $\mathrm{emb} : V \times U \to V \otimes U$ is defined by $\mathrm{emb}((v,u)) := v \otimes u$. Elements of $V \otimes U$ are often called tensors.

For example, the element $(a \cdot x + 0 \cdot z, y) - a(x,y) - 0 \cdot (z,y) \in Ident(V,U)$. Therefore, $(ax) \otimes y = a(x \otimes y)$, $\forall a \in \mathbb{F}$. In particular, $0 \otimes y = 0$.

The following statement may be considered a property characterizing the tensor product of two vector spaces.

Proposition 1.2. *Suppose that V, U and W are vector spaces over a field $\mathbb{F}$, and a map $f : V \times U \to W$ satisfies the following conditions:*

$$\begin{aligned} f(ax+bz,y) &= af(x,y) + bf(z,y), \\ f(x,ay+bt) &= af(x,y) + bf(x,t), \quad \forall x,z \in V, \forall y,t \in U, \forall a,b \in \mathbb{F}. \end{aligned} \tag{1.5}$$

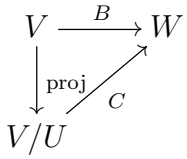
Then there exists a unique map $A \in \mathrm{Hom}(V \otimes U, W)$ such that $A \circ \mathrm{emb} = f$.

$$\begin{array}{ccc} V \times U & \xrightarrow{\mathrm{emb}} & V \otimes U \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle A} & \\ W & & \end{array}$$

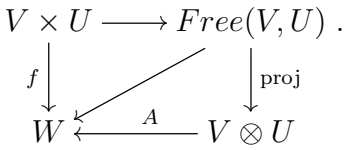
Date: August 20, 2025.

To prove Proposition 1.2, we first establish the following lemma.

Lemma 1.3. *Suppose that V, W are vector spaces, U is a subspace of V, denote by* $\mathrm{proj} : V \to V/U$ *the canonical projection, and $B \in \mathrm{Hom}(V, W)$. The existence of a map $C \in \mathrm{Hom}(V/U, W)$ such that $C \circ \mathrm{proj} = B$ is equivalent to the condition $U \subseteq \ker(B)$.*



Returning to the proof of Proposition 1.2, note that the diagram defines a map $g : Free(V, U) \to W$. We may then apply Lemma 1.3 to the lower left triangle of the diagram



Proposition 1.4. *Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V, $\{u_1, \ldots, u_m\}$ is a basis of U. Then*

$$\{v_i \otimes u_j \,|\, 1 \le i \le n, 1 \le j \le m\} \tag{1.6}$$

is a basis of $V \otimes U$.

The result of Proposition 1.2 can be used to prove the linear independence of the set in (1.6).

Proposition 1.5. *Suppose that V and U are finite-dimensional vector spaces. Define $\varphi : V^* \otimes U \to \mathrm{Hom}(V, U)$ by*

$$\varphi(f \otimes u)(v) := f(v)u, \quad \forall v \in V, u \in U, f \in V^*. \tag{1.7}$$

The map φ is well-defined and $\varphi \in Iso(V^* \otimes U, \mathrm{Hom}(V, U))$.

Note that in Proposition 1.5, the map is defined only on the tensors of the form $v \otimes u$, which is common. This definition follows the approach of Proposition 1.2.

An important corollary follows as a special case of Proposition 1.5.

Proposition 1.6. *Suppose that V is a finite-dimensional vector space. Then the vector spaces $V \otimes V^*$ and $\mathrm{End}(V)$ are canonically isomorphic.*

Definition 1.7. Suppose that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(U)$. Define the tensor product of linear operators as $A \otimes B \in \mathrm{End}(V \otimes U)$ by the formula

$$(A \otimes B)(v \otimes u) := (Av) \otimes (Bu) \tag{1.8}$$

on the tensors of the form $v \otimes u$ and then extend by linearity. By definition, we set

$$A^{\otimes k} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}} \tag{1.9}$$

Exercise 1.8. Show that $A \otimes B$ is well-defined.

Exercise 1.9. Let $A, B \in \mathrm{End}(V)$. Is it always true that

$$(AB)^{\otimes k} = ((A)^{\otimes k})((B)^{\otimes k})? \tag{1.10}$$

Exercise 1.10. Suppose that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \dots, v_n\}$ is a basis of V, $\{u_1, \dots, u_m\}$ is a basis of U, $A \in \mathrm{End}(V)$, its matrix in the basis $\{v_1, \dots, v_n\}$ is

$$(a_i^j)_{1 \le i,j \le n}, \tag{1.11}$$

$B \in \mathrm{End}(U)$, its matrix in the basis and $\{u_1, \dots, u_m\}$ is

$$(b_i^j)_{1 \le i,j \le m}. \tag{1.12}$$

Find the matrix of $A \otimes B$ in the basis

$$\{v_1 \otimes u_1, v_1 \otimes u_2, \dots, v_1 \otimes u_m, v_2 \otimes u_1, \dots, v_n \otimes u_m\}. \tag{1.13}$$

2. Symmetric Group

Definition 2.1. A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if it is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called a symmetric group and is denoted by S_n. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n). \end{pmatrix} \tag{2.1}$$

A more common notation is a cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{2.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{2.3}$$

It can be checked that this relation is indeed an equivalence relation. For each equivalence class that consists of, say, m elements we pick a representative j and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of ρ. If $m = 1$, we usually omit the corresponding one-element cycle. For example, the following cycle notation represents the permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{2.4}$$

It can be shown that cycle notation is well-defined. A cycle of length 2 is called a transposition.

"For example, the product of the two permutations in cycle notation is:

$$((1, 2)(3, 4, 5))((2, 3)(1, 5)) = (1, 3)(2, 4, 5). \tag{2.5}$$

Exercise 2.2. Show that S_n forms a group under composition and find its order.

One may compare the following statement with the change of basis for linear operators.

Exercise 2.3. Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$, the following equality holds

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{2.6}$$

where the LHS is the composition of three permutations such that the second permutation is a cycle of length k.

Exercise 2.4. Suppose that $\rho \in S_n$. Consider a map $\psi_\rho : S_n \to S_n$ defined by

$$\psi_\rho(\sigma) := \rho\sigma\rho^{-1}. \tag{2.7}$$

Show that ψ_ρ is an automorphism of the group S_n.

Exercise 2.5. Show that every permutation $\rho \in S_n$ can be expressed as a product of transpositions.

Proposition 2.6. *For every positive integer n there exists a unique map* $\mathrm{sgn} : S_n \to \{1, -1\}$ *such that*

(1) $\mathrm{sgn}((i,j)) = -1, \quad \forall 1 \le i < j \le n;$
(2) $\mathrm{sgn}(\rho\sigma) = \mathrm{sgn}(\rho)\mathrm{sgn}(\sigma).$

The value $\mathrm{sgn}(\rho)$ *is called the sign of the permutation ρ.*

To prove this proposition, one may consider the ratio

$$\frac{\prod_{1\le i<j\le n}(x_{\rho(i)} - x_{\rho(j)})}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{2.8}$$

3. Symmetric and Exterior Powers

Definition 3.1. Let V be a vector space over $\mathbb{F}$. Define the k-th symmetric and exterior powers of V, denoted respectively by $S^k(V)$ and $\Lambda^k(V)$, as subspaces of

$$V^{\otimes k} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}} \tag{3.1}$$

given by

$$S^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V\rangle, \tag{3.2}$$

$$\Lambda^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V\rangle. \tag{3.3}$$

We denote

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k := \frac{1}{k!} \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)}. \tag{3.4}$$

Exercise 3.2. Suppose that $\dim V = n$. Find $\dim S^k(V)$ and $\dim \Lambda^k(V)$.

Definition 3.3. Suppose that $A \in \mathrm{End}(V)$. One may show that

$$\forall z \in S^k(V) \text{ we get that } A^{\otimes k}(z) \in S^k(V), \tag{3.5}$$

$$\forall z \in \Lambda^k(V) \text{ we get that } A^{\otimes k}(z) \in \Lambda^k(V). \tag{3.6}$$

We are going to denote by $S^k(A) \in \mathrm{End}(S^k(V))$ and $\Lambda^k(A) \in \mathrm{End}(\Lambda^k(V))$ the restrictions of $A^{\otimes k}$ to the subspaces $S^k(V)$ and $\Lambda^k(V)$, respectively.

4. Determinant

Definition 4.1. Suppose that V is a finite-dimensional vector space, $A \in \mathrm{End}(V)$. Note that $\dim(\Lambda^{\dim V}(V)) = 1$. Therefore, $\Lambda^{\dim V}(A)$ is a multiple of the identity map. We define the determinant $\det(A)$ of the linear operator A by

$$\boxed{\Lambda^{\dim V}(A) = \det(A) \cdot \mathrm{Id}.} \tag{4.1}$$

Proposition 4.2. *Suppose that* $\dim V < \infty$ *and* $A, B \in \mathrm{End}(V)$. *Then*

$$\boxed{\det(AB) = \det(A) \cdot \det(B).} \tag{4.2}$$

Proposition 4.3. *Suppose that* $\dim V = n < \infty$ *and* $A \in \mathrm{End}(V)$ *has a matrix* $(a_i^j)_{1 \le i,j \le n}$ *in some basis. Then*

$$\boxed{\det(A) = \sum_{\rho \in S_n} \mathrm{sgn}(\rho) \prod_{j=1}^{n} a^j_{\rho(j)}.} \tag{4.3}$$

Chapters from the textbooks relevant to the lecture:

- Roman, pp. 363-378, 393-395.