LINEAR ALGEBRA, PART 5. COMPLEXIFICATION.

Please consider the proof of every proposition in this lecture to be an exercise.

1. Complexification.

Definition 1.1. Let V be a real vector space. Let us consider the field $\mathbb{C}$ as a vector space over $\mathbb{R}$. The tensor product $\mathbb{C}\otimes V$ is also a real vector space. We will introduce a complex vector space structure on $\mathbb{C}\otimes V$ by setting

$$\lambda\cdot(z\otimes v):=(\lambda z\otimes v),\quad \forall\lambda,z\in\mathbb{C},\forall v\in V. \tag{1.1}$$

Here, we define multiplication by a fixed λ on the set of pairs $\mathbb{C}\times V$. Note that it is bilinear, and we then extend this map to the tensor product $\mathbb{C}\otimes V$ using its universal property. We also define addition to be the same as in the tensor product $\mathbb{C}\otimes V$. The complex vector space that we have obtained is called the complexification of the real vector space V and is denoted by $V^{\mathbb{C}}$. Also, define

$$V_0^{\mathbb{C}}:=\{1\otimes v\mid v\in V\}. \tag{1.2}$$

For every subspace U of the vector space V, denote by $U^{\mathbb{C}}$ the following subspace of $V^{\mathbb{C}}$:

$$U^{\mathbb{C}}:=\{z\otimes u\mid z\in\mathbb{C},u\in U\} \tag{1.3}$$

Exercise 1.2. Let V be a real vector space. Show that $\forall v\in V^{\mathbb{C}}\ \exists u,w\in V$ such that

$$v=1\otimes u+i\otimes w. \tag{1.4}$$

Exercise 1.3. Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V<\infty$, then the dimension of $V^{\mathbb{C}}$ over C equals $\dim V$. What can be said about $V_0^{\mathbb{C}}$ and its dimension?

Definition 1.4. Suppose that V is a real vector space and $A\in\mathrm{End}(V)$. We define the complexification $A^{\mathbb{C}}\in\mathrm{End}(V^{\mathbb{C}})$ of the linear operator A by

$$A^{\mathbb{C}}(z\otimes v):=z\otimes(Av). \tag{1.5}$$

Note that

$$\forall v\in V_0^{\mathbb{C}}\quad A^{\mathbb{C}}v\in V_0^{\mathbb{C}}, \tag{1.6}$$

and the restriction of $A^{\mathbb{C}}$ to $V_0^{\mathbb{C}}$ coincides with A. Thus, we can view $A^{\mathbb{C}}$ as an extension of the operator A defined on $V_0^{\mathbb{C}}$. We are going to exploit this viewpoint a lot in the next section.

Example 1.5. Suppose that V is a real vector space, $\alpha=\{v_1,\dots,v_n\}$ is a basis of V, and $A\in\mathrm{End}(V)$. Then the set $\beta=\{1\otimes v_1,\dots,1\otimes v_n\}$ is a basis of $V^{\mathbb{C}}$, and the corresponding matrices coincide:

$$[A^{\mathbb{C}}]_\beta=[A]_\alpha. \tag{1.7}$$

Proposition 1.6. *Suppose that V is a real vector space and $A\in\mathrm{End}(V)$. Then the following equality holds for the characteristic polynomials*

$$\chi_{A^{\mathbb{C}}}(x)=\chi_A(x),\quad \forall x\in\mathbb{R}. \tag{1.8}$$

Note that $\chi_A\in\mathbb{R}[x]$. So Proposition 1.6 states that the characteristic polynomial of the linear operator $A^{\mathbb{C}}$ has real coefficients and is equal to χ_A.

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Example 1.7. Suppose that V is a complex vector space, $\gamma = \{v_1, \dots, v_n\}$ is a basis of V, $D \in \mathrm{End}(V)$, and

$$[D]_\gamma = (a_k^l + ib_k^l)_{1 \le k,l \le n} = A + iB, \quad A, B \in \mathrm{Mat}(n \times n, \mathbb{R}). \tag{1.9}$$

Now let us consider V as a vector space over $\mathbb{R}$, denoted by $V^{\mathbb{R}}$ to avoid confusion. Then the map D is a linear operator of $V^{\mathbb{R}}$; denote it by $D^{\mathbb{R}}$. Then $\delta = \{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ is a basis of $V^{\mathbb{R}}$. We get that

$$[D^{\mathbb{R}}]_\delta = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{1.10}$$

Proposition 1.8. *Let V be a finite-dimensional real vector space. Define* Conj $\in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *by*

$$\mathrm{Conj}(z \otimes v) := \overline{z} \otimes v. \tag{1.11}$$

Then:

a) $\mathrm{Conj}^2 = \mathrm{Id}$*;*

b) $\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ A^{\mathbb{C}} = A^{\mathbb{C}} \circ \mathrm{Conj}$*;*

c) We get that $\frac{1}{2}(\mathrm{Id} + \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and it is a projection operator onto* $V_0^{\mathbb{C}}$*;*

d) We get that $\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and*

$$\mathrm{im}\left(\frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})\right) = V_0^{\mathbb{C}}; \tag{1.12}$$

e)

$$\mathrm{im}\left(\frac{1}{2}(\mathrm{Id} + \mathrm{Conj})\right) \oplus \mathrm{im}\left(\frac{1}{2}(\mathrm{Id} - \mathrm{Conj})\right) = (V^{\mathbb{C}})^{\mathbb{R}}. \tag{1.13}$$

Proposition 1.9. *Suppose that V is a real finite-dimensional vector space and $A \in \mathrm{End}(V)$. Denote by V^λ a generalized eigenspace of $A^{\mathbb{C}}$. Then* $\mathrm{Conj}(V^\lambda) = V^{\overline{\lambda}}$.

Proposition 1.10. *Suppose that V is a real finite-dimensional vector space and W is a subspace of $V^{\mathbb{C}}$. Then the following conditions are equivalent:*

$$(\mathrm{Conj}(W) = W) \quad \Longleftrightarrow \quad (\textit{There exists a subspace } U \subset V \textit{ such that } W = U^{\mathbb{C}}). \tag{1.14}$$

Chapters from the textbooks relevant to this lecture:

- Roman, pp. 82-83.