LINEAR ALGEBRA, PART 6. INNER PRODUCT SPACES

Please consider the proof of every proposition in this lecture to be an exercise.

1. Definition of an Inner Product

Definition 1.1. Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. An inner product on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F} \tag{1.1}$$

with the following properties:

1) For $\mathbb{F} = \mathbb{C}$ (Conjugate symmetry)

$$\langle y, x \rangle = \overline{\langle x, y \rangle}, \quad \forall x, y \in V; \tag{1.2}$$

For $\mathbb{F} = \mathbb{R}$ (Symmetry)

$$\langle y, x \rangle = \langle x, y \rangle, \quad \forall x, y \in V; \tag{1.3}$$

2) (Linearity in the first entry)

$$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle, \quad \forall x, y, z \in V, \forall \lambda, \mu \in \mathbb{F}; \tag{1.4}$$

3) (Positive definiteness) For all $x \in V$,

$$\langle x, x \rangle \geq 0 \quad \text{and } (\langle x, x \rangle = 0) \quad \Longleftrightarrow \quad (x = 0). \tag{1.5}$$

A real (or complex) vector space V, together with an inner product, is called a real (or complex) inner product space.

It follows from the first property that $\langle x, x \rangle \in \mathbb{R}$, $\forall x \in V$. So the inequality in the third property makes sense even when $\mathbb{F} = \mathbb{C}$.

Question 1.2. Does the identity

$$\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle \tag{1.6}$$

hold $\forall x, y, z \in V, \forall \lambda, \mu \in \mathbb{F}$?

Example 1.3. Define an inner product on $\mathbb{C}^n$ by

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle := \sum_{j=1}^{n} x_j \overline{y_j}. \tag{1.7}$$

Proposition 1.4 (Cauchy-Schwarz inequality)**.** *Suppose that V is an inner product space. Then*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle, \quad \forall x, y \in V. \tag{1.8}$$

We may consider the expression $|x - \lambda y|^2$ to show this result.

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Definition 1.5. Suppose V is a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then a function $|\cdot| : V \to \mathbb{R}$ is called a norm if the following conditions hold:

1)
$$|x| \geq 0 \text{ and } ((|x| = 0) \iff (x = 0)); \tag{1.9}$$

2)
$$|\lambda x| = |\lambda| \cdot |x|, \quad \forall x \in V, \forall \lambda \in \mathbb{F}; \tag{1.10}$$

3) (The triangle inequality)
$$|x + y| \leq |x| + |y|, \quad \forall x, y \in V. \tag{1.11}$$

A vector space with a norm is called a normed space.

Note that, in the second property, we denote by the same symbol both the absolute value of an element of the field and the norm of a vector.

Proposition 1.6. *For an inner product space, V the formula*
$$|x| := \sqrt{\langle x, x \rangle} \tag{1.12}$$
defines a norm in V.

Proposition 1.7 (The polarization identities)**.** *a) Let V be an inner product vector space over $\mathbb{R}$. We claim that the following identity holds*
$$\langle x, y \rangle = \frac{1}{4} \left(|x + y|^2 - |x - y|^2 \right), \quad \forall x, y \in V. \tag{1.13}$$

b) Let V be an inner product vector space over $\mathbb{C}$, then the following identity holds
$$\langle x, y \rangle = \frac{1}{4} \left(|x + y|^2 - |x - y|^2 + i|x + iy|^2 - i|x - iy|^2 \right), \quad \forall x, y \in V. \tag{1.14}$$

2. Orthogonality

Definition 2.1. Suppose that V is an inner product space. Two vectors x and y are said to be orthogonal if $\langle x, y \rangle = 0$. For an arbitrary subset S of V, we define its orthogonal complement by
$$S^{\perp} := \{x \in V \mid \forall y \in S \, \langle x, y \rangle = 0\}. \tag{2.1}$$
A subset α of V is called orthogonal if $\forall x, y \in \alpha$, $x \neq y$ we have that $\langle x, y \rangle = 0$. A subset α of V is called orthonormal if it is orthogonal and $|x| = 1$, $\forall x \in \alpha$.

Exercise 2.2. Is it true that every orthogonal set of nonzero vectors is linearly independent?

Proposition 2.3. *Suppose that $\{v_1, \ldots, v_k\}$ is an orthogonal set of nonzero vectors in an inner product space V, $w \notin \operatorname{span}\langle v_1, \ldots, v_k \rangle$. Then there exists a unique vector v_{k+1}, up to multiplication by a scalar, satisfying the following properties:*

a) The set $\{v_1, \ldots, v_k, v_{k+1}\}$ is an orthogonal set of nonzero vectors;

b) $v_{k+1} \in \operatorname{span}\{v_1, \ldots, v_k, w\}$. This vector v_{k+1} can be expressed as
$$v_{k+1} := w - \sum_{j=1}^{k} \frac{\langle w, v_j \rangle}{\langle v_j, v_j \rangle} v_j. \tag{2.2}$$

Proposition 2.4 (Gram-Schmidt orthogonalization process)**.** *Suppose that* $\{w_1, \dots, w_m\}$ *is a linearly independent set in an inner product space* V*. Define the set* $\{v_1, \dots, v_m\}$ *by*

$$v_1 := w_1, \quad v_{k+1} := w_{k+1} - \sum_{j=1}^{k} \frac{\langle w_{k+1}, v_j \rangle}{\langle v_j, v_j \rangle} v_j, \quad k = 2, \dots, m. \tag{2.3}$$

Then the set $\{v_1, \dots, v_m\}$ *is an orthogonal set of nonzero vectors and*

$$\operatorname{span}\{v_1, \dots, v_k\} = \operatorname{span}\{w_1, \dots, w_k\}, \quad \forall 1 \le k \le m. \tag{2.4}$$

Proposition 2.5. *In a finite-dimensional inner product space, every orthogonal set is a subset of an orthogonal basis. In a finite-dimensional inner product space, every orthonormal set is a subset of an orthonormal basis.*

3. Orthogonal Projection

Proposition 3.1. *Suppose* V *is a finite-dimensional inner product space, and* U *is its subspace. Then*

$$V = U \oplus U^{\perp} \tag{3.1}$$

and

$$(U^{\perp})^{\perp} = U. \tag{3.2}$$

Proposition 3.2. *Suppose that* $\{v_1, \dots, v_n\}$ *is an orthonormal basis of an inner product space* V*. Then for every vector* $x \in V$*, the following decomposition holds:*

$$x = \sum_{j=1}^{n} \langle x, v_j \rangle v_j. \tag{3.3}$$

Definition 3.3. Suppose that $S = \{v_1, \dots, v_k\}$ is an orthonormal set of an inner product space V and x is an arbitrary vector in V. The vector

$$\widehat{x} := \sum_{j=1}^{k} \langle x, v_j \rangle v_j \tag{3.4}$$

is called the Fourier expansion of x with respect to S. Each coefficient $\langle x, v_j \rangle$ is called the Fourier coefficient.

Proposition 3.4. *Suppose that* $S = \{v_1, \dots, v_k\}$ *is an orthonormal set of an inner product space* V*,* $U = \operatorname{span}(S)$ *and* x *is an arbitrary vector in* V*. Then*

a) $\widehat{x}$ *is a unique vector of* U *such that* $x - \widehat{x}$ *is orthogonal to* U*;*

b)

$$\forall u \in U \quad |x - \widehat{x}| \le |x - u|; \tag{3.5}$$

c) (Bessel's inequality)

$$\forall x \in V \quad |\widehat{x}| \le |x|. \tag{3.6}$$

Proposition 3.5. *Suppose that* $S = \{v_1, \dots, v_k\}$ *is an orthonormal set of an inner product space* V *and* $U = \operatorname{span}(S)$*, then the following four conditions are equivalent:*

(1) S *is an orthonormal basis of* V*;*

(2) $\forall x \in V$ *we have* $\widehat{x} = x$*;*

(3) *(Bessel's identity)* $\forall x \in V$ *we get the equality* $|\widehat{x}| = |x|$;
(4) *(Parseval's identity)* $\forall x, y \in V$ *we have that*

$$\langle x, y\rangle = \sum_{j=1}^{k} \langle x, v_j\rangle \overline{\langle y, v_j\rangle}. \tag{3.7}$$

4. Riesz Representation Theorem

Definition 4.1. Suppose that V and W are complex vector spaces. A map $A : V \to W$ is called conjugate linear if $\forall x, y \in V, \forall \lambda, \mu \in \mathbb{C}$ the following equality holds

$$A(\lambda x + \mu y) = \overline{\lambda} A(x) + \overline{\mu} A(y). \tag{4.1}$$

Proposition 4.2 (Riesz Representation Theorem)**.** *Suppose that* V *is a finite-dimensional inner product space. Define the map* $A : V \to V^*$ *by*

$$A(x)(y) := \langle y, x\rangle. \tag{4.2}$$

We claim that A *is a conjugate linear bijective map (a conjugate isomorphism).*

Definition 4.3. Consider the map $A : V \to V^*$ defined in Proposition 4.2. For a covector $f \in V^*$ the preimage $A^{-1} f$ is called its Riesz vector and is denoted by R_f.

Chapters from the textbooks relevant to the lecture:

- Roman, Ch.9.
- Friedberg, Insel, Spence, Sections 6.1, 6.2;
- Axler, Ch.6;