LINEAR ALGEBRA, PART 7. OPERATORS IN INNER PRODUCT SPACES

Please consider the proof of each proposition in the lecture to be an exercise. In this lecture, the notation $\mathbb{F}$ stands for a field $\mathbb{R}$ or $\mathbb{C}$.

1. Adjoint of an operator

Definition 1.1. Suppose V is a finite-dimensional inner product space and $A \in \mathrm{End}(V)$. An operator is called the adjoint of A and is denoted by A^* if the following equality holds

$$\langle A^*x, y\rangle = \langle x, Ay\rangle. \tag{1.1}$$

Exercise 1.2. Let V be a finite-dimensional inner product space. Show that, for every $A \in \mathrm{End}(V)$, there exists a unique adjoint $A^* \in \mathrm{End}(V)$.

Proposition 1.3. *Let V be a finite-dimensional inner product space. For every $A, B \in \mathrm{End}(V)$*

(1) $(A+B)^* = A^* + B^*$;
(2) $(\lambda A)^* = \overline{\lambda} A^*, \quad \forall \lambda \in \mathbb{F}$;
(3) $(A^*)^* = A$;
(4) $(AB)^* = B^*A^*$;
(5) *if A is invertible then* $(A^{-1})^* = (A^*)^{-1}$.

Proposition 1.4. *Suppose that V is a finite-dimensional inner product space, U is a subspace of V, and $A \in \mathrm{End}(V)$. If the subspace U is A-invariant, then its orthogonal complement $U^\perp$ is also A^*-invariant.*

Proposition 1.5. *Suppose V is a finite-dimensional inner product space and $A \in \mathrm{End}(V)$. Then*

(1) $(\ker(A))^\perp = \mathrm{im}(A^*)$;
(2) $(\ker(A^*))^\perp = \mathrm{im}(A)$;
(3)
$$(A \text{ is surjective}) \iff (A^* \text{ is injective}); \tag{1.2}$$
(4)
$$(A \text{ is injective}) \iff (A^* \text{ is surjective}). \tag{1.3}$$

Exercise 1.6. Let V be a finite-dimensional inner product space. Is it true that for every $A \in \mathrm{End}(V)$ we have that

$$\ker(A) = \ker(A^*A) \text{ and } \mathrm{im}(A) = \mathrm{im}(AA^*)? \tag{1.4}$$

2. Normal operators.

Definition 2.1. Let V be an inner product space. A linear operator $A \in \mathrm{End}(V)$ is called normal if $AA^* = A^*A$.

Proposition 2.2. *Suppose V is a finite-dimensional inner product space and α is an orthonormal basis. Then*

$$(A \in \mathrm{End}(V) \text{ is normal}) \iff ([A]_\alpha \cdot [A^*]_\alpha = [A^*]_\alpha \cdot [A]_\alpha). \tag{2.1}$$

Proposition 2.3. *Let $A \in \mathrm{End}(V)$ be a normal operator. Then*

(1) *$p(A)$ is also normal for every $p \in \mathbb{F}[x]$;*
(2) $|Ax| = |A^*x| \quad \forall x \in V$;
(3) $\ker(A) = \ker(A^*)$;

Date: September 14, 2025.

(4) $\ker(A^k) = \ker(A), \quad \forall k \geq 1$;
(5) *the minimal polynomial for A is a product of distinct irreducible polynomials.*

Proposition 2.4. *Suppose V is a finite-dimensional complex inner product space, $A \in \mathrm{End}(V)$ is a normal operator. Then A and A^* have a common eigenvector.*

We are ready to formulate a spectral theorem for normal operators in a finite-dimensional complex inner product space. We will add several equivalent ways to state this result.

Proposition 2.5. *Let V be a finite-dimensional complex inner product space. Then the following conditions are equivalent:*

(1) *$A \in \mathrm{End}(V)$ is a normal operator;*
(2) *there exists an orthonormal basis α such that the matrix $[A]_\alpha$ is diagonal;*
(3) *there exists an orthonormal basis of V consisting of eigenvectors of A;*
(4)

$$V = \bigoplus_{j=1}^{k} V_{\lambda_j}, \tag{2.2}$$

where $\lambda_1, \ldots, \lambda_k$ are all eigenvalues of A and V_λ is the eigenspace corresponding to the eigenvalue λ. The subspaces V_{λ_r} and V_{λ_s} are orthogonal if $r \neq s$;

(5)

$$A = \sum_{j=1}^{k} \lambda_j P_{V_{\lambda_j}}, \tag{2.3}$$

where we denote the orthogonal projection onto a subspace U by P_U.

One possible approach to the proof is to use Propositions 2.4 and 1.4. Alternatively, we may use Proposition 2.3.

We are going to consider the real case in the next section.

3. Complexification of an inner product space

Definition 3.1. Let V be a real inner product space. Define an inner product on $V^{\mathbb{C}}$ by

$$\langle 1 \otimes x_1 + i \otimes y_1, 1 \otimes x_2 + i \otimes y_2 \rangle_{\mathbb{C}} := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + i(\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle). \tag{3.1}$$

Exercise 3.2. Let V be a real inner product space. Show that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ defines an inner product in the complex vector space $V^{\mathbb{C}}$. We will take it as a default inner product in the complexification $V^{\mathbb{C}}$.

Exercise 3.3. Suppose that V is a finite-dimensional real inner product space and $A \in \mathrm{End}(V)$. Show that

$$(A^{\mathbb{C}})^* = (A^*)^{\mathbb{C}}. \tag{3.2}$$

Proposition 3.4. *Suppose that V is a finite-dimensional real inner product space and $A \in \mathrm{End}(V)$ is a normal operator. As usual, denote by $R(a, b)$ the following matrix*

$$R(a, b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{3.3}$$

Then there exists an orthonormal basis β of V such that

$$[A]_\beta = \begin{pmatrix} \mu_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mu_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R(a_1,b_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R(a_2,b_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R(a_q,b_q) \end{pmatrix} \tag{3.4}$$

for some, not necessarily distinct, $\mu_1, \mu_2, \dots, \mu_p, a_1, \dots, a_q \in \mathbb{R}$, $b_1, \dots, b_q \in \mathbb{R}_+$.

4. Isometries.

Definition 4.1. An isomorphism A between two inner product spaces V and W over the same field $\mathbb{R}$ or $\mathbb{C}$ is called an isometry if

$$\langle Ax, Ay\rangle_W = \langle x, y\rangle_V, \quad \forall x, y \in V. \tag{4.1}$$

Proposition 4.2. *Suppose that V and W are finite-dimensional inner product spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and $\dim V = \dim W$. Then there exists an isometry between V and W.*

5. Unitary operators

Definition 5.1. Let V be a complex inner product space. A linear operator A is called unitary if

$$\langle Ax, Ay\rangle = \langle x, y\rangle, \quad \forall x, y \in V. \tag{5.1}$$

Question 5.2. What can we conclude about the invertibility and the inverse of a unitary operator in a finite-dimensional inner product space?

Proposition 5.3. *A is a unitary operator if and only if $AA^* = A^*A = \mathrm{Id}$.*

Proposition 5.4. *Suppose that V is a complex inner product space, $\dim V = n$, and α is an orthonormal basis in V. Then*

$$(A \text{ is unitary}) \iff (\text{The columns(rows) of } [A]_\alpha \text{ form an orthonormal basis in } \mathbb{C}^n). \tag{5.2}$$

The spectral theorem for normal operators implies the following result for unitary operators.

Proposition 5.5. *For every unitary operator A in a finite-dimensional complex inner product space V there exists a basis α in which the matrix of A is diagonal:*

$$[A]_\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \quad \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C},\ |\lambda_j| = 1, 1 \le j \le n. \tag{5.3}$$

The analogues of unitary operators for the real case are called orthogonal.

Definition 5.6. Let V be a real inner product space. A linear operator A is called orthogonal if

$$\langle Ax, Ay\rangle = \langle x, y\rangle, \quad \forall x, y \in V. \tag{5.4}$$

Proposition 5.7. *Suppose V is a real inner product space, $\dim V = n$, and α is an orthonormal basis in V. Then*

$$(A \text{ is orthogonal}) \iff (\text{The columns(rows) of } [A]_\alpha \text{ form an orthonormal basis in } \mathbb{R}^n). \tag{5.5}$$

Definition 5.8. For $\psi \in \mathbb{R}$ denote by R_ψ the following matrix

$$R_\psi := \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix}. \tag{5.6}$$

Proposition 5.9. *For every orthogonal operator A in a finite-dimensional real inner product space V there exists a basis α in which*

$$[A]_\alpha = \begin{pmatrix} R_{\psi_1} & 0 & \dots & 0 & 0 & 0 \\ 0 & R_{\psi_2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{\psi_r} & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathrm{Id}_p & 0 \\ 0 & 0 & \dots & 0 & 0 & -\mathrm{Id}_q \end{pmatrix}, \quad p, q, r \geq 0,\ \psi_1, \psi_2, \dots, \psi_r \in \mathbb{R}, \tag{5.7}$$

and Id_p denotes the identity $p \times p$ matrix. The sizes of the zero matrices in the off-diagonal entries vary depending on the sizes of the corresponding blocks on the main diagonal.

Note that $\dim V = 2r + p + q$ in this case.

Also, notice that

$$R_\pi := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{5.8}$$

Suppose A is orthogonal and $\det(A) = 1$. Recall that $\det(R_\phi) = 1$, $\forall \phi \in \mathbb{R}$. Hence, we conclude that $\dim V$ is even, and we can write all diagonal blocks in the equality (5.7) as 2×2 blocks R_ϕ.

6. Special classes of normal operators

In this section, we will get spectral theorems for four more classes of operators. They can be easily deduced from the spectral theorems for normal operators in complex and real inner product spaces.

Definition 6.1. Let V be a complex inner product space. A linear operator A is called self-adjoint or Hermitian if $A^* = A$.

Proposition 6.2. *For every Hermitian operator A in a finite-dimensional complex inner product space V, there exists a basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \tag{6.1}$$

Definition 6.3. Let V be a real inner product space. A linear operator A is called symmetric if $A^* = A$.

Proposition 6.4. *For every symmetric operator A in a finite-dimensional real inner product space V there exists a basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \tag{6.2}$$

Definition 6.5. Let V be a complex inner product space. A linear operator A is called skew-Hermitian if $A^* = -A$.

Proposition 6.6. *For every skew-Hermitian operator A in a finite-dimensional complex inner product space V there exists a basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} i\mu_1 & 0 & \dots & 0 \\ 0 & i\mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & i\mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \tag{6.3}$$

Definition 6.7. Let V be a real inner product space. A linear operator A is called skew-symmetric if $A^* = -A$.

Proposition 6.8. *For every skew-symmetric operator A in a finite-dimensional real inner product space V there exists a basis α in which*

$$[A]_\alpha = \begin{pmatrix} R(0,b_1) & 0 & \dots & 0 & 0 \\ 0 & R(0.b_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & R(0.b_r) & 0 \\ 0 & 0 & \dots & 0 & 0_p \end{pmatrix}, \quad p.r \geq 0,\ b_1, b_2, \dots, b_r \in \mathbb{R}, \tag{6.4}$$

and 0_p stands for the zero $p \times p$ matrix.

7. Matrix interpretations of spectral theorems

Definition 7.1. We have introduced several classes of linear operators in inner product spaces - normal, unitary, orthogonal, Hermitian (self-adjoint), symmetric, skew-Hermitian, and skew-symmetric.

For each of those classes we will define a corresponding class of matrices via a similar rule. For instance, if $A \in \mathrm{End}(V)$ is a unitary operator, a basis α is orthonormal then the matrix $[A]_\alpha$ is called unitary. Or, we can state this condition differently:

A matrix $B \in \mathrm{Mat}(n \times n, \mathbb{C})$ is called unitary iff $(\overline{B})^t \cdot B = \mathrm{Id}$.

This definition allows us to reformulate each of the spectral theorems in terms of the matrices. We will write one example.

Proposition 7.2. *Let $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ be a normal complex matrix. Then there exist unitary matrix U and a diagonal matrix D such that*

$$A = UDU^{-1}. \tag{7.1}$$

Chapters from the textbooks relevant to the lecture:

- Roman, Ch. 10.
- Friedberg, Insel, Spence, Sections 6.3–6.6;
- Axler, Sections 7A–7D;