"Basic Representation Theory", Fall 2025.

Course description. Welcome to the Basic Representation Theory course! Representation Theory studies how groups or other algebraic objects may act on vector spaces. Representations are a key ingredient in various models in Mathematical Physics, Number Theory, Algebraic Combinatorics, and other areas of mathematics. The first part of the course covers the main results of the classical theory of complex representations of finite groups. The second part introduces matrix Lie groups, Lie algebras and their representations. We will consider only finite-dimensional representations.

We have structured the course to provide you with opportunities to prove all theorems collaboratively with your peers. Several days before each class, you will receive lecture notes containing definitions, examples, and propositions that we will need to prove. In class, we will discuss your approaches, comments, and any questions you may have.

The Program of the Course.

1. Alongside the discussion of representations we are going to recall some (multi)linear algebra concepts and theorems: Jordan canonical form theorem, bilinear forms, inner product spaces, spectral theorem for unitary operators, tensor products of vector spaces, symmetric and exterior algebras.
2. Examples of representations. Direct sum of representations. Invariant subspaces. Subrepresentations. Irreducible representations.
3. Morphisms of representations. Schur's lemma.
4. Complete reducibility. Maschke's theorem.
5. Uniqueness of the decomposition of a completely reducible representation into a sum of irreducible ones.
6. Tensor product of representations.
7. Characters of representations, their properties. Conjugacy classes.
8. Group algebra. Central functions. Regular representation.
9. Orthogonality of complex matrix elements.
10. Group algebra of a finite group as a product of matrix algebras, corresponding to irreducible representations.
11. Number of irreducible complex characters. Orthogonality relations.
12. Examples of complex character tables: abelian groups, dihedral group D_n, groups S_3, S_4, A_4. Decomposition of tensor products of irreducible representations.
13. * Representations of symmetric groups, Young diagrams.
14. Examples of Lie groups. Definition of a matrix Lie group. Connected Lie groups.
15. Covering of SO(3,R) by SU(2).
16. A Lie algebra is the tangent space of a Lie group at the identity element.
17. Definition of an abstract Lie algebra. Adjoint representation.
18. Representations of the Lie algebra sl(2,C).
19. The connection between representations of Lie groups and Lie algebras. Polynomial representation of the Lie group sl(2,C).

20. Tensor product of representations of Lie algebras. Clebsch-Gordan decomposition for the Lie algebra sl(2,C).
21. Compact groups and their representations. Peter-Weyl theorem.
22. Highest weight sl(3,C)-modules.
23. * Complex simple Lie algebras and their irreducible representations.

* - if time allows

Recommended Textbooks.

1. W. Fulton, J. Harris, Representation Theory. A First Course.

2. E.B. Vinberg, Linear Representations of Groups.

3. G.James, M.Liebeck, Representations and Characters of Groups. (for items 1-5 of the program)

4. B.C. Hall, Lie Groups, Lie Algebras, and Representations. (for items 6-10 of the program)

Grading Rules. You should submit solutions to at least three problems by each Saturday. The due time is one hour before the first class. You will get 1 point for every correct solution but no more than three points for each week. Additional bonus points may be awarded for the shortest and most elegant solutions or solutions for the most complicated exercises. Both the midterm and the final exam are oral exams, open books, and open notes. The midterm contains 5 problems for 3 points each, and the final – 5 problems for 4 points each. Also, you get one bonus point every time you are the first to correct the professor's mistake/mistype during the lecture. The grading table has the following form:

A+	90%
A	85%
A-	80%
B+	75%
B	70%
B-	65%
C+	55%
C	50%
C-	45%
D	35%

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